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Definitions in Mathematics: What do High School Students Know?

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Introduction

Mathematics is a tool for problem-solving. It is a means of modeling. It is its own language. Mathematicians are innovators. They are problem-posers. They are solution-seekers. Yet much of our mathematics education consists of procedures and algorithms that can be learned by rote memorization. How do we bridge the gap from the mindless regurgitation of mathematical procedures to the ability to use mathematics to solve challenging problems? The key is to understand the mathematics. Students who understand *why* the procedures work can then apply these procedures in new and different situations. Understanding, is that all? How do we, as math educators, ensure the understanding of *all* our students?

As previously stated, mathematics is its own language. What makes it confusing is that it shares many of the same words, and similar-sounding words, that we use in common English. Upon introducing myself as a mathematics major to a new acquaintance, I was presented with the following string of puns:

"Did you hear about Jack and Jill? They make acute couple."

"Did you try the Boston Cream Pi in the dining hall today? It was delicious."

"I have these friends who are twins. It's so hard to differentiate them."

After a few more, another friend who claimed to be less than amused added:

"If you don't stop with these math jokes, I'm going to divide you by half!"

This final comment caught my attention for reasons other than the dreadful brilliance of the first few puns. Dividing your friend *by* half would really mean doubling him. Surely he meant to say that he wished to divide his friend *in* half. This goes to show how significantly the mathematical meaning changes by switching one little word.

While these examples purposefully implied the English meaning of the mathematical homonym, students who are not fluent with the different, and often more precise, mathematical definitions can be confused by mathematical vocabulary. With this in mind, I decided to investigate the role of definitions in a high school mathematics class. More specifically, I was curious to investigate the following questions: Can students define the mathematical concepts they are studying? What role do definitions play in student understanding of the mathematics?

Based on my own personal experience, I know now that I have gotten through some math classes by following patterns and procedures without a strong understanding of why the mathematics works. It was not until later, when I was asked to explicitly define concepts and could not, that I even realized that I had not understood these concepts as well as I had thought. Although I was successful in the classroom and on tests, I was not able to apply the mathematics I was learning to new situations that required a deep understanding of why the mathematics works. This ability to apply what we have learned in mathematics is what we should hope to take from any mathematics class. Therefore, the aforementioned questions about student understanding in relation to their ability to write definitions are valid and important inquiries.

Research Questions

Through my research, I hoped to answer the following questions about definitions in mathematics:

1. Are students able to write their own precise definitions for various mathematical terms in a high school pre-calculus class?
2. Does the ability to write precise definitions translate to students being able to successfully do mathematics? Related, does the *in*ability to write precise definitions translate to students being *un*able to successfully do mathematics?

I chose to investigate these questions in a high school pre-calculus class because at this point in their high school mathematics careers, students have studied functions quite a bit. They are past the introductory phase of pre-algebra and should be developing a deeper conceptual understanding of a function and its domain and range. After all, this is a pre-calculus class which should be preparing the students to work with limits and integrals of functions in calculus, work which must build upon a solid foundation of understanding functions. Additionally, for many students, this is the last mathematics class they will take in high school. Since much of the high school mathematics curriculum is centered on algebra and the idea of functions, I wanted to know what students graduating from high school understand about functions. Even for those students who do not go on to take a calculus course, I should expect that they have developed a strong understanding of functions since this may be the last time they work with functions. Yet, in my own experience as a college junior, I was having doubts about the domains of inverse functions in conjunction with function composition. This revealed a hole in my own understanding of functions. Therefore, it is apparent that even though students *should* have a strong grasp on functions by the time they graduate from high school, there are often gaps in conceptual understanding.

Background Research

There are contrasting opinions about the importance of definitions in doing mathematics. Mathematical terms have precise and specialized definitions, more so than their common English homonyms. For example, in common English, the word 'similar' is an adjective meaning "marked by correspondence or resemblance" (WolframAlpha, 2013); one could look at two chocolate cakes and say, "Hey! Those look similar" because they are both chocolate. On the other hand, the more precise mathematical definition of 'similar' is "a property of two figures whose corresponding angles are all equal and whose distances are all increased by the same ratio" (WolframAlpha, 2013); in the previous example, the cakes would only be similar if they were the same shape and their corresponding side lengths were proportionate. Definitions are often used in writing proofs, but nowhere else in the high school curriculum are students expected to write the definitions that they were meant to have learned.

One belief is that definitions are necessary for doing mathematics. In his paper, "Learning the Language of Mathematics", Robert E. Jamison defines what makes a definition 'good' and argues the importance of teaching students how to write good definitions. He claims that only "once students understand HOW things are said, they can better understand WHAT is being said, and only then do they have a chance to know WHY it is said (author's emphasis)" (Jamison, 2000, p. 1). Jamison defines a definition as "a *concise* statement of the *basic* properties of an object or concept which *unambiguously identify* that object or concept" (p. 4). Definitions that are complete and concise, and that state only and all the necessary information to distinguish the object or concept, are considered 'good'. Jamison believes that these 'good' definitions are essential for doing mathematics because students who understand the definitions can begin to learn why the mathematics of those definitions works.

Alternatively, Shlomo Vinner believes that formal definitions are not essential to doing mathematics. He considers the ideas of "concept image" and "concept definition" in mathematics. Vinner (1990) claims that neither students nor mathematicians use the formal concept definitions in order to do mathematics; instead they access their evolving concept image, which is a visual representation or collection of experiences related to the given term (p. 68). Ideally, a student's concept image and concept definition will influence each other so that the student has a more complete understanding of the mathematics. For a depiction of the interaction between one's concept image and the concept definition, see Figure 1.

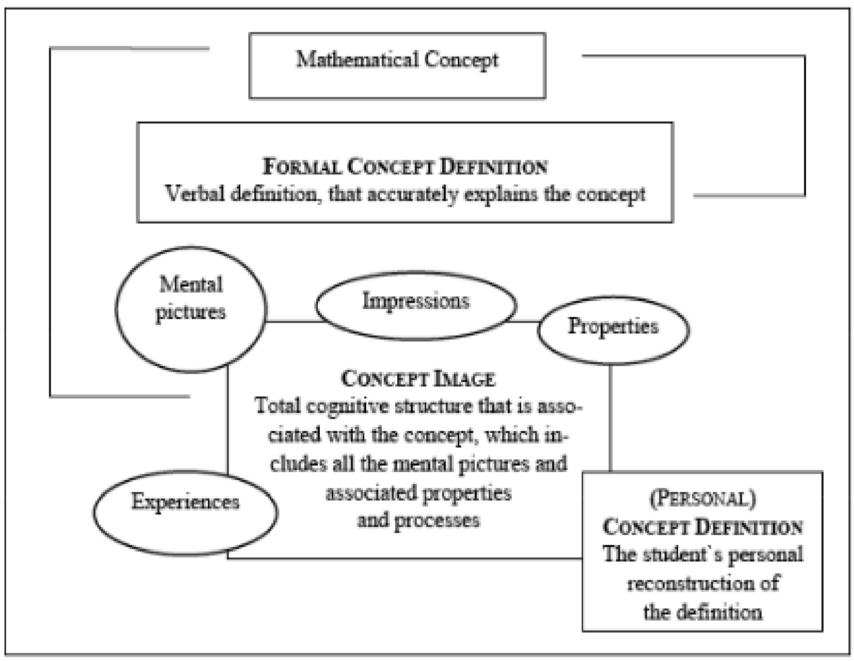

Figure 1: Concept image and concept definition map

There have been many studies looking into the use of concept maps by students and teachers in learning and teaching mathematics. A concept map is a graphic organizer used to illustrate connections made from a given topic or term; in a sense, it can be used to depict one's concept image. Concept maps may be created by teachers or students. A concept map created by a teacher is often used to illustrate connections between topics that the teacher wants the students to learn, while a concept map created by a student may be used to assess a student's understanding of the topic; periodically redrawing the concept map may illuminate students' evolving understanding. To name a few, Williams (1998), Schmittau (2004), Dogan-Dunlap, Torres and Chen (2005), and Grevholm (2008) have all studied the use of concept maps in the mathematics classroom. In general, they found that connections within students' concept maps were very limited at the beginning of the study; the concept maps often included only a single example with no interpretation, generalization, or connection to other mathematical topics. However, as students learned more, their concept maps grew and evolved. Grevholm (2008) gives the example of Lina, whose concept map of 'function' had only a dozen connections to start out, and had nearly thirty connections just over a year later (p. 6). Grevholm describes this as a typical example that depicts how a student's concept image expands over time.

These studies illustrate the importance of understanding mathematical vocabulary. It may be that the formal definitions are necessary for students to understand the concepts, or that a concept image rich with connections to other mathematical topics is more useful. Thus, I chose to ask students to write formal definitions as well as draw pictures and give examples of the concepts in order to probe their concept image.

Rationale

As previously noted, I chose to work with students in a pre-calculus class because the concept of functions was one with which the students should be quite familiar. To answer my research questions, I chose to have students define the words 'domain', 'range', and 'function' because these are concepts that are central to many areas of mathematics. They are also words that each have common English meanings that may influence how students think about that term in its mathematical context. The ideas of domain, range, and function are particularly important for modeling real-world situations and are the foundation of much higher-level mathematics that students may study. Additionally, as I was working with juniors and senior in a pre-calculus class, the topic of functions is one that they have been studying for four years since pre-algebra. Over those four years, I expected that students would have developed a good understanding of these concepts. Although students are not usually asked to explicitly state the definitions of mathematical vocabulary, the following standards describe what students should be able to do in relation to functions and communicating their understanding.

The Common Core State Standards in Mathematics (CCSSM, 2010) have a high school standard focused on Functions. These standards address students' abilities to interpret functions (IF), build functions (BF) and work with various categories of functions such as linear, quadratic, exponential and trigonometric functions.

<u>Common Core State Standards (CCSS) for Mathematical Content (pp. 68-70):</u>

- F-IF-1: Understand that a function from one set (called the domain) to another set (called the range) assigns to each element of the domain exactly one element of the range. If f is a function and x is an element of its domain, then $f(x)$ denotes the output of f corresponding to the input of x. The graph of f is the graph of the equation $y=f(x)$.

- F-IF-3: Recognize that sequences are functions, sometimes defined recursively, whose domain is a subset of the integers. For example, the Fibonacci sequence is defined recursively by $f(0) = f(1) = 1$, $f(n+1) = f(n) + f(n-1)$ for $n \geq 1$.
- F-BF-1c: Write a function that describes a relationship between two quantities.

  (+) Compose functions. For example, if $T(y)$ is the temperature in the atmosphere as a function of height and $h(t)$ is the height of a weather balloon as a function of time, then $T(h(x))$ is the temperature at the location of the weather balloon as a function of time.

The National Council of Teachers of Mathematics' (NCTM) *Principles and Standards for School Mathematics* (2000) includes the function concept within the Algebra Standard at the high school level. These standards suggest that by the completion of high school all students should be able to demonstrate the following:

<u>NCTM Algebra Standards (pp. 296-306):</u>

- Understand relations and functions and select, convert flexibly among, and use various representations for them.
- Use a variety of symbolic representations, including recursive and parametric equations, for functions and relations.

In addition to the content specific standards described above, the NCTM also includes standards that address the importance of students' abilities to communicate mathematically.

<u>NCTM Communication Standard (pp. 348-352):</u>

- Use the language of mathematics to express mathematical ideas precisely.

Though these mathematical standards do not require students to be able to write definitions, students are expected to understand various forms of functions and to use functions to model real-world situations. Additionally, students are expected to be able to communicate their mathematical knowledge, conjectures, and reasoning so that others may understand. In particular, I relied on their communication in my study in order to see what they really understood about functions.

Methodology

For this study, I worked with three students in a high school college-preparatory pre-calculus class. The students and their parents were given consent and assent forms to fill out. I intended to work with the classroom teacher to ensure a variety of academic levels in the three to five students with whom I would work. However, I only received three forms back from the students; thus, I worked with all three students, regardless of academic level within the class. See Appendix A for the Informed Consent form and Appendix B for the Student Assent form.

I planned on meeting with the three students as a group during the last half hour of one of their class periods. So that they did not miss any instruction, we ended up meeting for two fifteen-minute sessions on different days, as that was easier to schedule into their class period.

In the first meeting, the students were asked to describe in words, pictures, and examples the following terms: domain, range, and function. The purpose of this was to get an idea of their concept images for the given terms. Then they were asked to define in a complete, grammatically correct sentence, the same terms. They also rated their confidence in their definitions on a scale from 1 to 5, with 1 being "not at all confident" and 5 being "very confident". After completing these definitions individually, the students and I discussed their definitions and reviewed several textbook definitions. The textbooks used were their own pre-calculus text, Larson's eighth edition of *Precalculus* (2011), and Hass, Weir, and Thomas's *University Calculus* (2007). After this review, the students were given the opportunity to rewrite each definition as a group. I took detailed notes about what the students said throughout our discussion in order to supplement their written work. Due to time constraints, this concluded the first meeting with the students. The students' worksheets were collected.

The second meeting took place during the next class meeting. Their worksheets were re-distributed and the students were also given copies of the textbook definitions for the chosen vocabulary words. The given definitions were as follows:

The **domain** is the set of all inputs of the function *f*.

The **range** is the set of all outputs of the function *f*.

A **function** *f* is a relationship between domain and range such that each element *x* in the domain corresponds to exactly one element *y* in the range.

During our second meeting, students were asked to do some mathematics. First, they were asked to determine if a function existed that met the description of a given relation or graph. They were not asked to produce this function, but determine its existence. The next section asked students to identify whether or not the given equations were indeed functions. They were asked to explain why or why not, as well as to state the domain and range of each. Finally, students were given a word problem where they had to choose between two given composite functions which of these composite functions represented the given situation. I took an audio recording of this session in order to better complete my notes of what the students were saying as they talked through the problems with one another. For the student worksheet, see Appendix C.

I had originally planned on asking students to define two additional terms: 'inverse function' and 'function composition'. I had also hoped to spend more time creating definitions with the students, with the expectation that the definitions that students spent time in creating would be more memorable and more readily accessible when doing the mathematics. To determine how memorable those definitions actually would be, I planned on returning to the classroom several weeks after the initial meetings to ask students to define the same words and to answer similar questions of existence, identity, and applications related to functions. Because of time constraints, this follow-up meeting did not take place.

I analyzed the data qualitatively by using the students' written work to outline each student's individual abilities. I referred to my notes from our discussions to supplement their written work and gain a more complete picture of their concept images. I looked for evidence of the students' use of the definitions in doing the mathematics problems. I compared the students' responses to each other, and I looked at how their responses compared to mathematically accepted definitions and solutions. After observing what each student appeared to understand, I made generalizations about the students' successes, and identified specific weaknesses and misconceptions in the students' understanding as their responses were compared to the mathematically accepted solutions.

In the next section, I provide profiles of the participating students. Each profile describes the student's responses to the written work, as well as the student's level of participation in the group discussion.

Student Responses

The students I worked with in this study were a mix of juniors and seniors in high school. I worked with two girls and one boy. The students were of average to above average ability. In general, the students were reluctant to participate in the group discussion. Although they wanted to work together to complete their worksheet, they hesitated in explaining aloud their thought processes even when asked to do so.

Student A

In the definitions section, Student A did not write definitions in complete sentences, nor did she rate her confidence in her descriptions of the terms. For *domain*, Student A wrote "x-axis" and bulleted "x – numbers you put in". For *range*, she wrote "(-∞,∞)", "y-axis", and bulleted "values that come out of function". For *function*, Student A bulleted "plug something in to get something else out" and wrote "f(x) = y". She also drew a picture of the graph of a function, taking the shape of a function such as $y=-x^3$ or $y=-\tan x$. On the drawing of this graph, Student A drew dashed vertical lines and told me that this was to show that the graph is the graph of a function because it passes the vertical line test. To contrast, she drew a picture of a graph, appearing to be $y=\pm\sqrt{x}$, that does not pass the vertical line test. See Figure 2 below.

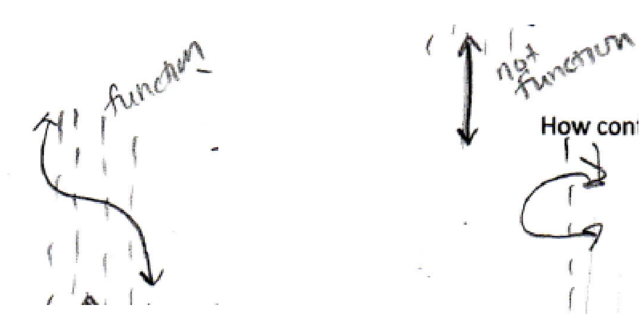

Figure 2: Student A's example and non-example of a function.

Is there a function such that…

a) …each positive number corresponds to 1, each negative number corresponds to -1 and 0 corresponds to 0?
b) ... $f(0) = 1$, $f(1) = 1$, and $f(n) = f(n-1) + f(n-2)$ for $n \geq 2$?
c) ...its graph is the following?

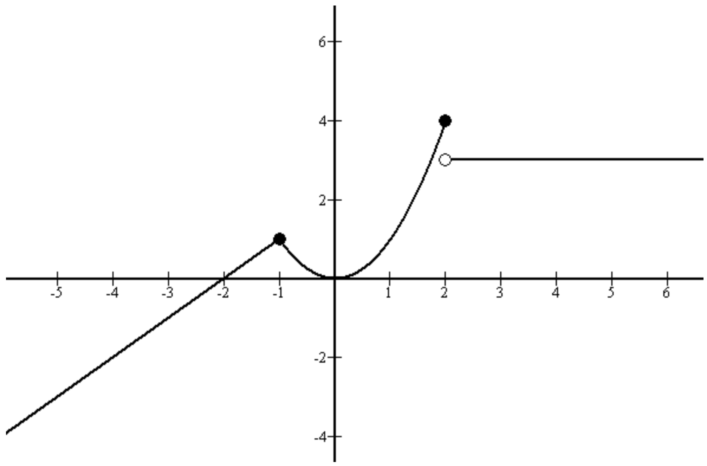

Figure 3: Questions of existence from student worksheet

To answer the questions of existence (see Figure 3 above), Student A incorrectly responded that there did not exist a function for condition (a) that each positive number corresponds to 1, each negative number corresponds to -1, and 0 corresponds to 0. She left part (b), which described the Fibonacci sequence as a function, blank. She correctly wrote for part (c) that the piece-wise graph including a hole and a jump was a function and she demonstrated that by drawing eight vertical lines through various parts of the graph, again calling on the vertical line test to make her decision.

Which of the following are functions? Explain why or why not. State the domain and range of each.

a. $y = 3sinx$
b. $y - ½ = 4x^3$

Figure 4: Questions of identification from student worksheet

The first question (see Figure 4 above) asking the student to identify whether a given relation represented a function was $y=3sinx$. Student A correctly identified this as a function; however, her reasoning was questionable. She wrote, "Yes, it's a function because sine is a steady wave that passes the vertical line test." The questionable word in this explanation is 'steady.' It does not offer any insight on whether the relation or its graph represents a function, but it suggests that Student A, like many students, views continuity as a necessary condition for a function. On the other hand, this student correctly recognized the piece-wise graph as a function, which is clearly not continuous. This goes to show the importance of using precise terminology, as 'steady' has an ambiguous meaning as used here. The second identification question was $y-½=4x^3$. Student A isolated y, but then simply wrote "No". She gave no insight as to why she (incorrectly) determined this was not a function. Student A did not state the domain and range for either function in this section.

> You work forty hours a week at a furniture store. You receive a \$220 weekly salary, plus a 3% commission on sales over \$5000. Assume that you sell enough this week to get the commission. Given the functions $f(x) = 0.03x$ and $g(x) = x - 5000$, which of $(f \circ g)(x)$ and $(g \circ f)(x)$ represents your commission? (http://www.purplemath.com/modules/fcncomp5.htm)

Figure 5: Composition of functions application problem from student worksheet

Finally, with the application question dealing with function composition (see Figure 5 above), Student A initially wrote that the composite function $g(f(x))$ would represent the commission. After discussing her answer with the other two students, she agreed that $f(g(x))$ would correctly model the commission.

Overall, Student A demonstrated proficiency in using the vertical line test to determine whether the graph of a given relation represented a function. However, she was reluctant to provide further reasoning, either verbal or written, to support her thinking.

Student B

For the definitions, Student B described *domain* as “the value of x” and “the given parameters of a[n] equation for the x axis”. Student B was the only student to write a complete sentence giving the definition of the term. He wrote, “The domain is the value of x, and it also states the parameters of the x axis in an equation.” Student B described *range* as “the value of y” and “the given parameters of y in the function”. His definitions was, “The range is the value of y, and it gives the parameters of the y axis.” Student B self-identified as very confident in his definitions of *domain* and *range*.

In describing the term *function,* Student B drew a table with the first column labeled ‘x’ and the second column labeled ‘y’. He also drew the coordinate plane with a vertical line to demonstrate a graph that was not a function, and a coordinate plane with a horizontal line to demonstrate a graph that was a function. He did not label these graphs as examples and non-examples, but told me so in our discussion. For the true example of a function, he drew in a vertical line and told me that the graph passed the vertical line test. In addition, this student gave an example of an equation that is a function: “x+3=y”. Finally, Student B’s definition was “A function is a given equation where inputs are put in and outputs are solved.” He rated his confidence level as a “3,” indicating that he was neither confident nor unconfident that his definition was correct.

On the existence page (refer to Figure 3), Student B only responded (incorrectly) to the first question (a) saying that there is no function such that each positive number corresponds to 1, each negative number corresponds to -1, and 0 corresponds to 0. The recursively defined function and the graph of a function were left untouched.

For the first identification question (refer to Figure 4), Student B wrote, "Yes this is the function of sin[e] of x because of the trig form plus it also can pass the line test." Student B drew a graph of this function, labeling the points *3* and -*3* on the y-axis to show the amplitude. He drew vertical lines through the graph presumably to demonstrate that the function passed the vertical line test. For the second identification question dealing with a cubic function, Student B seemed to be unclear about how to graph the function, as he drew the coordinate plane, but no graph. His response to whether or not the cubic equation was a function was "Nah". Student B did not state the domain and range of either function.

For the application question (refer to Figure 5), Student B recognized *f(g(x))* as the composite function representing the commission. However, he wrote "f(gx)", which is incorrect notation. He explained, "plug in g into the x" and wrote "0.03x-5000". This, again, is inaccurate notation missing a set of parentheses. It should read '0.03(x-5000)'. Whether Student B understood the parentheses are necessary is unclear because he did not have to solve this equation for any value of *x*.

In general, Student B favored visual methods such as using the vertical line test to determine if the graph of a given relation represented a function. He seemed less motivated to answer some of the questions, as evidenced by his leaving several of them completely blank. It is unclear whether he left them untouched because he did not know how to approach the problem, or that he simply did not want to put forth the effort in attempting to figure out the answer.

Student C

Student C described *domain* as “what you can put in to get a correct answer or output” and labeled this “x”. She also said, “I think of an input output table” and drew an example of an input-output table that would represent the function $y=2x$ (see Figure 6 below). Student C’s definition was “A domain would be an input.” She indicated moderate confidence in the accuracy of her definition. Student C described the *range* as “y” and wrote the definition “A range would be the area or span of numbers that the answer could be in.” This definition illustrates some confusion between the mathematical and common English definitions of the word ‘range’. Student C rated her confidence level as a “3,” indicating that she was neither confident nor unconfident in her definition. Student C’s description of *function* included a graph of a function appearing to be $y=x^2$. Her definition was “A function is what you need to know to get the answer correct.” She indicated that she was moderately not confident (level “2”) that her definition was correct. Student C did not identify the domain and range of either of the functions.

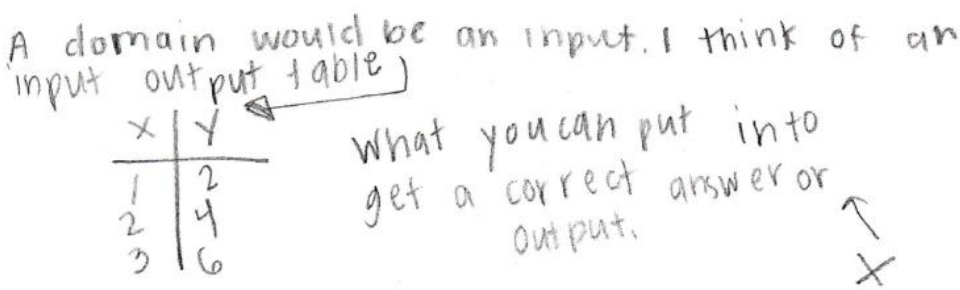

Figure 6: Student C's definition of 'domain' featuring an input-output table

For the existence section (refer to Figure 3), the Student C answered (incorrectly) without explanation that there was no function in part (a) such that each positive number corresponds to 1, each negative number corresponds to -1, and 0 corresponds to 0. She correctly identified, but with no explanation, that the recursively defined function in part (b) was a function. Given the graph in part (c), Student C recognized that it was the graph of a function, explaining “yes because of the vertical line test”.

For the identification questions (refer to Figure 4), Student C recognized *y=3sinx* as a function, explaining “it is a function because there can be multiple inputs” and “passes vertical line test”. When I asked her to clarify what she meant by “multiple inputs,” she was unable to phrase it any differently. She did not recognize the cubic equation as a function, writing “I don’t think so because x couldn’t serve as multiple inputs”. She also told me that after solving the equation for *y*, she was not sure what happened with the constant term at the end.

For the function composition application question (refer to Figure 5), Student C wrote both “$(f \circ g)(x)$” and “$(g \circ f)(x)$”. Underneath she wrote the correct expression to model the commission, “0.03(x-5000)” and stated in words that her model was the composite function *f(g(x))*.

Overall, Student C, like her classmates, used the vertical line test to determine whether the graph of a given relation represented a function. Similarly to her classmates, she was also uncomfortable explaining aloud her thinking. Student C expressed some confusion between a common English definition and the mathematical definition of ‘range.’ Student C was the first participant to represent a function in a different form, not as a graph or an equation, but as a table of values.

Group Work

After individually writing definitions, the students and I reviewed their definitions and the textbook definitions. I asked each student to read aloud his or her definition. The students were given the opportunity to ask the presenter to clarify his or her definition. Then I read each textbook definition twice, and the students were given the chance to talk with each other to clarify the definitions or to ask me questions about the definitions. After review of their definitions and textbook definitions, the students worked together to rewrite a definition for *function*. Their (incomplete) definition read, "A function is a relationship between domain and range."

After their individual work on the math problems of existence, identification, and application, the students compared their answers with each other. Group work focused on the function composition application problem since the students disagreed on which composite function, $f(g(x))$ or $g(f(x))$, would represent the commission. Students B and C were able to convince Student A that $f(g(x))$ correctly represented the commission by explicitly writing out each composite function, $f(g(x))=0.03(x-5000)$ and $g(f(x))=0.03x-5000$, at which point Student A quickly saw that $f(g(x))$ was the composite function representing the commission.

Although the students wanted to work together, and asked me multiple times during the individual part if it was time to work together yet, they were reluctant to explain their thinking verbally. They were willing to share their answers, but hesitant to reveal why they believed that those were the answers.

Overall, the students' written and verbal responses when completing the worksheet indicated a preference for visual representations and examples that were, presumably, meant to be generalized to descriptions the concepts. In particular, their verbal contributions to the discussion demonstrated reluctance to explain their reasoning in words.

General Findings & Analysis

In general, the students' definitions were lacking. Most "definitions" were not true formal definitions written in complete sentences that included all the defining characteristics. Instead, students wrote bulleted descriptions of the given terms. These descriptions, though not definitions, are still useful for looking into students' concept images. All of the students drew graphs of functions as an example of a function. This importance of a visual representation of function was further supported by the students' use of the vertical line test in determining whether the graph of a given equation was a function or not.

The students excelled in determining whether a given graph was the graph of a function. They all knew how to use the vertical line test, which requires that a vertical line drawn anywhere on the graph pass through no more than one point of the graph of the equation. At a deeper level of understanding, one should know why the vertical line test is sufficient for determining whether a graph is the graph of a function. A vertical line passing through only one point shows that for any value along the x-axis (any point in the domain), there is only one output value. It is unclear whether the students possessed this deeper understanding of why the vertical line test works; recall their collaborative definition: "A function is a relationship between domain and range". Because their definition of 'function' lacked this defining characteristic about each input value producing exactly one output value, I cannot be certain that the students truly understood the necessity of this attribute or the reasoning for the validity of the vertical line test.

There were some misconceptions evident in the students' attempts to define the given terms. For example, *range*, which has multiple mathematical meanings in addition to its common English meaning, seemed to cause confusion for one of the students. Wolfram-Alpha, a self-described computational knowledge engine, is an online resource that defines mathematical concepts and completes mathematical computations. It lists eighteen definitions of the word 'range', ten of which are nouns with the final eight being verbs. One such definition is the following: "(2) *noun,* the limits within which something can be effective" (WolframAlpha, 2013). In its mathematical sense, Wolfram-Alpha defines 'range' as "(7) *noun,* the set of values of the dependent variable for which a function is defined" (WolframAlpha, 2013). With such varied definitions, it is not surprising that Student C's mathematical definition of 'range' was influenced by the common meanings. Recall that Student C wrote, "A range would be the area or span of numbers that the answer could be in." She further identified "y" as the variable representing a value in the range; thus it is apparent that she associated 'range' with the output values. However, the phrasing of her definition seems to combine the ideas of Wolfram-Alpha's definitions (2) and the mathematical (7).

While many "definitions" were lists of descriptions, I found that the definitions that were written in complete sentences were incomplete; they conveyed all true information, but they excluded defining characteristics of the term that are necessary in its definition. In particular, the students' definition of 'function' as a relationship between domain and range was, of course, true. It was not an accurate and complete definition, though, because, as previously discussed, it was missing the distinguishing trait that each value in the domain is associated with exactly one value in the range. Attempting to use this incomplete definition to make a decision about whether or not a relationship is a function would be challenging. This demonstrates the necessity of more accurate formal definitions or more robust concept images, such as the students' reliance on the vertical line test rather than their written definitions.

The language that Student B used in his definitions was somewhat troublesome. Recall that he defined a function as "a given equation where inputs are put in and outputs are solved". Ignoring the fact that this definition is incomplete, lacking the defining characteristic of a function, the troubling phrase is "outputs are solved". 'Solve' is another term that has multiple meanings in both mathematics and common English. In mathematics, students are often asked to "solve an equation", which means that students are to find the roots or zeroes of the equation. This leads to specific x values from the domain that satisfy $f(x)=0$. Instead, Student B may have meant that "inputs are put in *and evaluated to produce an output*". The use of the term "evaluate" does not restrict the input values in the same way as the word "solve"; an equation can be evaluated at any point x in the domain.

Although the students were given copies of the textbook definitions, as well as having their own definitions available to use in answering the mathematical questions, none of the students referred back to their own or the textbook definitions when completing the worksheet. The students left some questions blank and misidentified equations of functions as not being functions, but they chose not, or forgot, to use their definitions to help them. The students self-identified as neither confident nor unconfident to moderately confident that their definitions of 'domain', 'range', and 'function' were correct. Therefore, it seems likely that the students did not feel it necessary that they refer back to their definitions when answering the questions because they felt that they knew and understood their definitions or perhaps relied more on other aspects of their concept images such as the graphical representations and the vertical line test.

In general, the students did not do well identifying the various equations as functions when there was not a graph to go with the equation. All students recognized the equation involving sine as a function, which should be expected as they were studying trigonometric functions at the time the interviews took place. However, the students did not recognize the cubic equation as a function, they did not recognize the piece-wise-defined function as a function, and most did not recognize the recursively-defined function as being a function. Student C, who identified the recursively-defined function as a function, offered no explanation, even when prompted, as to why she thought it was a function. Thus, it is uncertain whether she understood why it is a function or if she simply guessed.

Finally, despite the fact that the students as a whole felt moderately confident in their definitions of *domain* and *range*, none of the students stated the domain and range of the given functions in the identification section. They also asked questions about the composite function application question that make me doubt their grasp on the domain and range of functions. The students repeatedly told me that neither composite function could represent the commission because both functions would give a negative answer. They understood that in the context of this problem, a negative answer, meaning losing money, would not make sense. What the students did not realize was that x had to be large enough so that the output of the function was never negative. The domain of the function was restricted to the real numbers greater than 5000; in the context of the problem, this models the commission that was awarded on sales greater than $5000.

Overall, the students demonstrated a simple visual understanding of functions as evidenced by their repeated use of the vertical line test in determining whether the graph of a given relation represented a function. However, they did not interpret this as a necessary condition when they wrote their definition of ‘function.’ In general, the students were uncomfortable expressing the definitions and their thinking either verbally or in writing. They would give examples and draw graphs of functions, but they did not go further in talking about *why* and explaining their reasoning. They were reluctant to write definitions of the given terms in complete sentences. Finally, the students did not refer to their definitions or the textbook definitions when completing the mathematics problems.

Conclusions

My first research question asked whether or not students in a high school pre-calculus class are able to write precise definitions for mathematical concepts, specifically for *domain*, *range*, and *function*. I required that definitions be written in complete sentences since we speak in complete sentences when we communicate and explain our ideas to one another. I found that students were not able to write definitions in this way. They shared with me descriptions and examples, which form part of their concept images, but they were unable to express the concept definition. This may be evidence of a lack of understanding or that students did not want to commit something to paper that may be incorrect. It may also be that students could not write definitions for lack of practice; if students are not required to write definitions or explain on paper their solutions, it certainly would not be easy for them to do so in the short amount of time we had together.

My second question inquired about students' ability to do mathematics in relation to their ability to define the mathematics. I do not feel as though I gathered enough information in this study to gauge the students' mathematical abilities. Due to the condensed time in which this study took place, I did not get the chance to have students answer more questions about identifying functions and their domains and ranges, or about modeling real-world situations using functions. The students misidentified many of the examples that I was able to ask them about; however, I recognize that these examples, such as the recursively-defined function and the wordy description of a function, do not take the typical form of a function that these students are accustomed to seeing. While, according to state standards, they should be able to identify them as functions regardless, this recognition requires a deeper understanding of functions since the functions were presented in a new way.

Additionally, I found that the students did not attempt to use their written definitions or those from the textbook to help in solving the mathematics problems. This may indicate a disconnect between definitions and problem-solving in the traditional high school curriculum. In my own experience and my observations of various high school mathematics classes, I have not seen teachers show students how to use a definition. A definition is simply something to be presented at the beginning of a unit. This idea is comparable to Vinner's belief about students and mathematicians alike not using formal definitions, but accessing their concept images when doing mathematics.

This study is not meant to, nor can it, generalize its findings to all high school pre-calculus students. It is merely an example of three students' ideas about functions in an average-level pre-calculus class. The questions and observations in this study were limited by time; our meetings occurred during their class period, but it was important to me and their teacher that they did not miss any instructional time. It would have been beneficial to have seen the students answer additional questions about identifying functions given more time.

This study was also limited by the students' willingness to participate. Only three students agreed to participate in this study. Despite the fact that I did not get to choose, in discussion with the teacher, students to ensure a variety of abilities, the students I worked with were a mix of juniors and seniors, of students who made the honor roll and one that did not.

A further limitation of this study is the credibility of the students' responses to some of the questions about identifying functions. One student told me, and another agreed that they just guessed and really had no idea if it was a function or not. The students' reluctance to try to explain their thinking to me is likely due to the limited time in which we had to work together and their discomfort or inability in finding the words to communicate their thinking to me in an understandable way.

While this study did not look to claim that definitions are or are not essential to doing mathematics, it appears that the students did not use the definitions when answering the math questions. Therefore, as teachers, it may be important to model referring back to definitions to demonstrate to students that learning is not a one-time deal and then you are done; instead, it is a continuous process that involves evaluation and reevaluation of one's thoughts about the topic to be learned. Furthermore, teachers should help students acquire a more robust concept image. They should help students to make connections in the map of a student's concept image in order to deepen the student's understanding of the given term or concept, as this concept image seems to be what students refer to when they think about how to do mathematics. For example, it might have been helpful for these students if they understood the connections between the vertical line test and the function definition.

Though not a primary focus of the research questions, but of the much broader topic of language in mathematics, this study is another example of how the language of mathematics is very particular. However, language is also manmade. The same word can mean different things to different people, and not only because it has multiple accepted definitions. In mathematics, students may understand in their heads, but be unable to communicate well that understanding because they do not know the language to do so. The precise language of mathematics must be taught to all students in order for the communication necessary among groups of people to have meaning.

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Appendix A: Informed Consent Form

March 2013

Dear Parent(s) or Guardian(s),

My name is Julie Sacco and I am a senior undergraduate student in the Math Department at UNH and I am conducting a research project as part of my thesis to find out the role of definitions in a high school pre-calculus class. I am writing to invite your child to participate in this project. I plan to work with approximately 4-5 students in this study.

I will be working with students in a group setting. If you allow your child to participate in this study, your child will be asked to define various mathematical terms and solve several related problems. Audio recordings will be taken for students to explain their solutions. The total time commitment is expected to be approximately one hour, consisting of two half-hour meetings during your child's pre-calculus class, coordinated with Ms. St. Cyr so that your child does not miss instruction of new material. Neither you nor your child will receive any compensation to participate in this project.

The potential risks of your child participating in this study are minimal; I will minimize the possibility of recognition in my written thesis and presentation by using pseudonyms for all participants. I will maintain confidentiality of your child's responses by asking students not to talk about the study outside of the group. Although your child is not expected to receive any direct benefits from participating in this study, (s)he may benefit from the extra practice with definitions, leading to a better understanding of the material. Additionally, this study may be useful to current and future teachers in gauging the importance of mathematical definitions for student understanding.

Participation is strictly voluntary; your refusal to allow your child to participate will involve no prejudice, penalty, or loss of benefits to which you or your child would otherwise be entitled. Your child may refuse to answer any question. If you allow your child to participate in this project and your child wants to, and then either you change your mind or your child changes his/her mind, you may withdraw your child, or your child may withdraw, at any time during the study without penalty.

I seek to maintain the confidentiality of all data and records associated with your child's participation in this research; however, it is possible that other participants will repeat responses outside of the study. I will ask students not to share information from the study outside of the group and explain that this is to maintain confidentiality of their responses. I will keep all student work and audio recordings locked in Dr. Sharon McCrone's office (my thesis advisor); only Dr. McCrone and I will have access to the data. Audio recordings will be destroyed upon completion of this study. I will report the data using pseudonyms for all participants. The results will be used in writing my thesis and a presentation.

If you have any questions about this research project or would like more information before, during, or after the study, you may contact Julie Sacco at (603) 496-3017 or by email, jmn597@unh.edu or Dr. Sharon McCrone at (603) 862-3587 or by email, smy72@unh.edu. If you have questions about your child's rights as a research subject, you may contact Dr. Julie Simpson in UNH Research Integrity Services at 603-862-2003 or Julie.simpson@unh.edu to discuss them.

Please sign and return the following page indicating your choice no later than Tuesday, March 26. You may keep this letter for your records. Thank you for your consideration.

Sincerely,

Julie Sacco and Dr. Sharon McCrone

Parent/Guardian Informed Consent Form

(Students 17 years of age or younger)

I understand the purpose of this study is to investigate students' knowledge of mathematical definitions and the effect this has on how they do the related mathematics.

I understand that participation is voluntary and I/we or my child may choose to withdraw from the study at any time without penalty.

Yes, I, __________________consent/allow my child _______________________ to participate in this research project.

No, I, ____________________do not consent/allow my child _____________________ to participate in this research project.

______________________________ ________________________

Signature of Parent/Guardian Date

____

Printed Name(s) of Parent(s)/Guardian(s)

Appendix B: Student Assent Form

March 2013

Dear Student,

My name is Julie Sacco and I am a senior undergraduate student in the Math Department at UNH and I am conducting a research project as part of my thesis to find out the role of definitions in a high school pre-calculus class. I am writing to invite you to participate in this project. I plan to work with approximately 4-5 students in this study.

I will be working with students in a group setting. If you agree to participate in this study, you will be asked to define various mathematical terms and solve several related problems along with others in the group. Audio recordings will be taken for you to explain your solutions. The time total commitment is expected to be approximately one hour, consisting of two half-hour meetings during your pre-calculus class, coordinated with Ms. St. Cyr so that you do not miss instruction of new material. You will not receive any compensation to participate in this project.

The potential risks of participating in this study are minimal; I will minimize the possibility of recognition in my thesis and presentation by using pseudonyms for all participants. I will ask you and your fellow participants not to talk about the study outside of the group in order to maintain confidentiality of your responses. Although you are not expected to receive any direct benefits from participating in this study, you may benefit from the extra practice with definitions, leading to a better understanding of the material. Additionally, this study may be useful to current and future teachers in gauging the importance of mathematical definitions in student learning.

Participation is strictly voluntary; your refusal to participate will involve no prejudice, penalty, or loss of benefits to which you would otherwise be entitled. You may refuse to answer any question. If you agree participate in this project, and then you change your mind, you may withdraw, at any time during the study without penalty.

I seek to maintain the confidentiality of all data and records associated with your participation in this research; however, it is possible that other participants will repeat responses outside of the study. I will ask students not to share information from the study outside of the group and explain that this is to maintain confidentiality of your responses. I will keep all student work and audio recordings locked in Dr. Sharon McCrone's office (my thesis advisor); only Dr. McCrone and I will have access to the data. Audio recordings will be destroyed upon completion of this study. I will report the data using pseudonyms for all participants. The results will be used in writing my thesis and a presentation.

If you have any questions about this research project or would like more information before, during, or after the study, you may contact Julie Sacco at (603) 496-3017 or by email, jmn597@unh.edu or Dr. Sharon McCrone at (603) 862-3587 or by email, smy72@unh.edu. If you have questions about your rights as a research subject, you may contact Dr. Julie Simpson in UNH Research Integrity Services at 603-862-2003 or Julie.simpson@unh.edu to discuss them.

Please sign and return the following page indicating your choice. You may keep this letter for your records. Thank you for your consideration.

Sincerely,

Julie Sacco and Dr. Sharon McCrone

Student Assent Form

I understand the purpose of this study is to investigate students' knowledge of mathematical definitions and the effect this has on how they do the related mathematics.

I understand that my participation is voluntary and I may choose to withdraw from the study at any time without penalty.

Yes, I, ___________________ agree to participate in this research project.

No, I, ____________________do not agree to participate in this research project.

______________________________ ________________________

Signature of Student Date

____

Printed Name of Student

Appendix C: Student Worksheet

Describe the following terms using words, pictures, examples, or any other way you can think of. Then define the terms in complete sentences.

Domain

How confident are you that your definition is correct? Not at all confident ----> Very confident

1 2 3 4 5

Range

How confident are you that your definition is correct? Not at all confident ----> Very confident

1 2 3 4 5

Function

How confident are you that your definition is correct? Not at all confident ----> Very confident

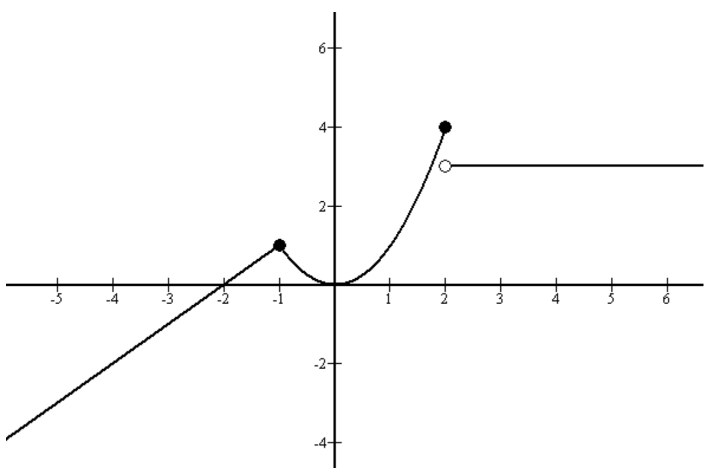
1 2 3 4 5

Is there a function such that…

a) …each positive number corresponds to 1, each negative number corresponds to -1 and 0 corresponds to 0?

b) ... $f(0) = 1$, $f(1) = 1$, and $f(n) = f(n-1) + f(n-2)$ for $n \geq 2$?

c) ...its graph is the following?



Please solve the following problems.

1. Which of the following are functions? Explain why or why not. State the domain and range of each. ~~Does each function have an inverse? If not, how might you restrict the domain so that it does have an inverse? Is there only one way to make this restriction?~~ [Note: Due to lack of time, I asked the students not to complete the questions about inverses.]
   a. $y = 3sinx$

   b. $y - ½ = 4x^3$

2. You work forty hours a week at a furniture store. You receive a $220 weekly salary, plus a 3% commission on sales over $5000. Assume that you sell enough this week to get the commission. Given the functions $f(x) = 0.03x$ and $g(x) = x - 5000$, which of $(f \circ g)(x)$ and $(g \circ f)(x)$ represents your commission? (http://www.purplemath.com/modules/fcncomp5.htm)