Physics experiences and calculus: How students use physics to construct meaningful conceptualizations of calculus concepts in an interdisciplinary calculus /physics course

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Physics experiences and calculus: How students use physics to construct meaningful conceptualizations of calculus concepts in an interdisciplinary calculus /physics course

Abstract
The purpose of this study was to investigate the manner by which students enrolled in an integrated Calculus/Physics course use their understanding of physics to inform their conceptualizations of calculus concepts. This study utilized a multiple case study design with analysis by and across cases. The cases represent eight first year students in the College of Engineering and Physical Sciences at the University of New Hampshire who enrolled in an integrated calculus/physics program. Data was gathered in a three-part process: (1) Semi-structured task-based interviews, (2) Participant-observation in the calculus/physics course, and (3) Obtaining copies of students’ in-class notes, in-class activities, homework assignments, and examinations.

A series of tasks designed to elicit information about students' conceptualizations of average rate of change, derivative, integral, and the Fundamental Theorem of Calculus were developed and pilot-tested by the researcher. To further corroborate the information gathered through the interview tasks, the students' examination, homework assignments, and in-class activities were reviewed. A description of each students' concept image was developed by analyzing the students' responses to interview tasks and triangulated with student-produced concept maps, observation of students in class, and students' homework, performance on examinations, and class work. A second layer of analysis resulted in the emergence of a classification scheme that describes how the students use physics to inform their conceptualization of calculus concepts, if at all. Finally, by searching individual student descriptions for patterns and similarities, a general description for the interactions between concept image and classification was proposed.

The results from this research investigation suggest that students frequently use physics concepts to construct meaningful conceptualizations of average rate of change. However, the students less frequently draw upon physics concepts to inform their conceptualizations of derivative and integral. The results from this research investigation also suggest that the students participating in this study possess richer conceptualizations of calculus concepts that what has previously been reported in the literature.

Hypotheses and questions for further investigation of students’ uses of physics concepts to inform their conceptualizations of calculus concepts are generated. Implications for teaching practice and curriculum development are suggested and discussed.

Keywords
Education, Mathematics

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PHYSICS EXPERIENCES AND CALCULUS: HOW STUDENTS USE
PHYSICS TO CONSTRUCT MEANINGFUL
CONCEPTUALIZATIONS OF CALCULUS CONCEPTS IN AN
INTERDISCIPLINARY CALCULUS/PHYSICS COURSE

BY

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M.S., Lehigh University, 1997

DISSERTATION

Submitted to the University of New Hampshire
in Partial Fulfillment of
the Requirements for the Degree of

Doctor of Philosophy

in

Mathematics Education

September, 2001
This dissertation has been examined and approved.

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July 20, 2001
Date
DEDICATION

This dissertation is dedicated to my family. You dared me to dream and then helped me fulfill my dreams. Your love and support has sustained me in countless ways throughout this journey. I am deeply grateful to have a family who gives so much and asks for so little in return.

To my grandparents, Bertha, John, Margaret, and Rocco: Although you are not able to witness this accomplishment, I know that you are sharing in the joy of my achievement. Your thoughts and prayers have guided and comforted me throughout this process. You have been pillars of unquestionable confidence in me.

To my parents, Angela and Joseph: My accomplishments are every bit as much yours as they are my own. I am deeply grateful to you for your constant encouragement, support, guidance, and love. You have fostered in me a love of learning and taught me that hard work pays off. You never questioned my dreams or doubted my ability to fulfill my goals, even when I doubted myself. You always supported my decisions and encouraged me during rough times. Your strength and love continue to be an essential part of my life.

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ABSTRACT

PHYSICS EXPERIENCES AND CALCULUS: HOW STUDENTS USE PHYSICS TO CONSTRUCT MEANINGFUL CONCEPTUALIZATIONS OF CALCULUS CONCEPTS IN AN INTERDISCIPLINARY CALCULUS/PHYSICS COURSE

by

Karen Ann Marrongelle

University of New Hampshire, September 2001

The purpose of this study was to investigate the manner by which students enrolled in an integrated Calculus/Physics course use their understanding of physics to inform their conceptualizations of calculus concepts. This study utilized a multiple case study design with analysis by and across cases. The cases represent eight first year students in the College of Engineering and Physical Sciences at the University of New Hampshire who enrolled in an integrated calculus/physics program. Data was gathered in a three-part process: (1) Semi-structured task-based interviews, (2) Participant-observation in the calculus/physics course, and (3) Obtaining copies of students’ in-class notes, in-class activities, homework assignments, and examinations.

A series of tasks designed to elicit information about students’ conceptualizations of average rate of change, derivative, integral, and the Fundamental Theorem of Calculus were developed and pilot-tested by the researcher. To further corroborate the information gathered through the interview tasks, the students’
examination, homework assignments, and in-class activities were reviewed. A description of each students’ concept image was developed by analyzing the students’ responses to interview tasks and triangulated with student-produced concept maps, observation of students in class, and students’ homework, performance on examinations, and class work. A second layer of analysis resulted in the emergence of a classification scheme that describes how the students use physics to inform their conceptualization of calculus concepts, if at all. Finally, by searching individual student descriptions for patterns and similarities, a general description for the interactions between concept image and classification was proposed.

The results from this research investigation suggest that students frequently use physics concepts to construct meaningful conceptualizations of average rate of change. However, the students less frequently draw upon physics concepts to inform their conceptualizations of derivative and integral. The results from this research investigation also suggest that the students participating in this study possess richer conceptualizations of calculus concepts that what has previously been reported in the literature.

Hypotheses and questions for further investigation of students’ uses of physics concepts to inform their conceptualizations of calculus concepts are generated. Implications for teaching practice and curriculum development are suggested and discussed.
CHAPTER I

THE RESEARCH PROBLEM

Introduction

Students' understanding of calculus concepts lays a foundation for their future study of advanced mathematics, science, and engineering courses. The idea of change—both how things change and the rate at which things change—plays a particularly important role in students' conceptualizations of calculus concepts. Students must understand the concept of rate of change in order to understand the derivative and differential equations. Furthermore, students must understand the idea of total change to understand the integral. Finally, students must understand the relationship between rate of change and total change in order to understand the relationship between derivatives and integrals outlined by the Fundamental Theorem of Calculus.

Undergraduate mathematics education researchers have called for more detailed investigations into the manner in which students develop conceptualizations of calculus concepts (Ferrini-Mundy & Graham, 1991; Hauger, 1997). One product of research that addresses student learning is its influence on teaching practices (DeCorte & Greer, et. al., 1996; Greeno, Collins et al., 1996). Specifically, understanding how students learn and come to know calculus concepts will help inform calculus teaching practice. Students enter the classroom with many experiences, both mathematical and non-mathematical, that will shape how they learn new mathematics concepts (Tall & Vinner, 1981). In particular, students may bring their experiences dealing with rate of change with them.
into the classroom and these experiences might influence how they develop an understanding of change.

In order to grasp abstract ideas of rate of change, students might rely on physical interpretations of the abstract notions of change (Nemirovsky, Tierney, & Ogonowski, 1992). Students may have encountered some of the underlying calculus concepts informally in everyday life, thus students will enter the calculus classroom with some intuition about concepts such as rate of change and derivative (Nemirovsky & Rubin, 1992; Nemirovsky & Noble, 1997). Furthermore, many students experience the mathematical concepts of average rate of change, derivative, and integral in physics classes as they study concepts such as motion, force, and electricity.

Physics, a typical introductory course for most engineering, science, and mathematics students, provides a context for which students can study change in a concrete setting. Studies have shown that mathematics understanding enhances the learning of physics concepts (Hudson & McIntire, 1977; Cohen, Hillman et al., 1978; Champagne, Klopfer et al., 1980), but have not examined how physics understanding affects the learning of calculus concepts. The present study attempts to develop an understanding of the nature of students' construction of calculus concepts and the factors influencing that construction. Specifically, I examined how students' understanding of physics concepts influences their understanding of calculus concepts.

Problem Statement

The motivation for the present research study grew out of my work as an evaluator of the Calculus/Physics program at the University of New Hampshire and a
subsequent review of the mathematics and physics education literatures. My work as an evaluator of the Calculus/Physics program uncovered differences between the manner in which Calculus/Physics students and traditional calculus students approached average rate of change and derivative tasks. The Calculus/Physics students tended to use physics terminology and concepts as they solved average rate of change and derivative tasks. The traditional calculus students, however, tended to rely on their memorization of mathematical formulas and processes as they solved average rate of change and derivative tasks. The Calculus/Physics students seemed to make more connections to their knowledge of physics as they solved the average rate of change and derivative tasks than the traditional calculus students. I looked to the mathematics and physics education literatures to help shed light on my finding that the Calculus/Physics students seemed to make meaningful connections between their calculus and physics understandings. The mathematics and physics education literatures, while substantial and rich in their respective areas, stand in isolation from one another. That is, the mathematics education literature, specifically literature addressing calculus learning and understanding has assumed a strictly mathematical perspective. Likewise, the physics education literature has assumed a strictly physical perspective. Additionally, the mathematics education literature to date has not addressed why some students do not possess conceptual understandings of calculus concepts. Rather, research has addressed questions concerning what students know and understand about calculus.

During the past few decades, researchers have begun to investigate the factors influencing student achievement in calculus and students’ understanding of calculus concepts. These studies focus on students’ understanding of concepts such as function
and limit (Dreyfus & Eisenberg, 1982; Davis & Vinner, 1986; Vinner, 1989; Williams, 1991; Carlson, 1997), students’ understanding of rate of change (Thompson, 1994b; Hauger, 1995; Bezuidenhout, 1998), and student understanding of the derivative and integral concepts (Orton, 1983a; 1983b; Ferrini-Mundy & Graham, 1994). Other studies focus on factors related to student achievement in calculus (Edge & Friedberg, 1984; Ferrini-Mundy & Gaudard, 1992) and the impact of alternative approaches to calculus instruction on students’ learning (Bookman & Friedman, 1994; Frid, 1994; Meel, 1998). Many of these studies have uncovered students’ misconceptions and have underscored the need for a more conceptual approach to teaching calculus concepts.

Research has shown that many students proficiently apply algorithms and procedures when asked to compute derivatives and integrals (Orton 1983a, 1983b; Ferrini-Mundy & Graham, 1994). These procedures include applying the power rule to differentiate polynomials, using the product, quotient, or chain rule to differentiate transcendental functions, and applying techniques such as integration by parts and trigonometric substitution to calculate definite integrals. However, some students are unable to discuss the conceptual underpinnings of the algorithms and procedures that they can so proficiently use. For example, Ferrini-Mundy and Graham (1994) show that a student who can apply the power rule and chain rule to take derivatives did not understand the relationship between the derivative of a function at a point and the slope of the tangent line of the function at that point.

Few research studies investigate the connections students make between calculus and other disciplines or calculus and the world. There is little research on students’ understanding of calculus concepts that addresses the real-world experiences that students
bring with them to the calculus classroom. Most investigations into students’ understanding of calculus concepts neglect the physical representation of the key concepts and primarily focus on numeric, algebraic, and graphical representations. The physical representations of calculus concepts link students’ experiences in the physical world with concepts in the calculus classroom. Neglecting students’ experiences with the physical world ignores a critical piece of students’ understanding of calculus concepts. Thus, there is a need for research that investigates the interplay between students’ experiences with the physical world and their understanding of calculus concepts.

Furthermore, the mathematics education literature to date stands in isolation from other bodies of educational research, specifically physics education research. One goal of the present research study is to extend the literature by examining student understanding of calculus concepts within a specific context.

The present study grew out of research done as part of an evaluation of an integrated Calculus/Physics program offered to first-year engineering and science students at the University of New Hampshire. As part of the evaluation of the calculus/physics program, I conducted clinical interviews with students enrolled in the calculus/physics class and students enrolled in the traditional calculus course. The purpose of the clinical interviews was to investigate similarities and differences between the calculus/physics students’ and traditional calculus students’ performances on average rate of change and derivative tasks.

A preliminary analysis of the clinical interview data uncovered differences in the manner in which the calculus/physics and traditional calculus students approached the interview tasks. In particular, I noticed that the calculus/physics students tended to use
physics terminology and concepts as they solved the average rate of change and derivative tasks, even though these tasks were presented to the students in a strictly mathematical context. For example, I presented the students with the graph of a function, \( f(x) \), and asked them to sketch the graph of the derivative function, \( f'(x) \). On the other hand, the traditional calculus students, who were either concurrently enrolled in a physics class or had previously taken a physics class, tended to use standard calculus vocabulary as they solved the average rate of change and derivative tasks. Furthermore, the calculus/physics students tended to use physics to help them resolve uncertainties when solving the problems and they used physics to justify their solutions to the problems. The traditional calculus students tended to rely on their memorization of procedures and algorithms as they solved the problems.

The differences between the calculus/physics students and the traditional calculus students mentioned above prompted me to search the literature for studies that addressed students’ understandings of calculus concepts and students’ use of physics to help them understand calculus concepts. While I found many studies that investigated students’ understanding of calculus concepts (Orton 1983a, 1983b, 1984; Bezuidenhout, 1999), I found few studies that explored the role of physics concepts in students’ understanding of calculus concepts.

The research conducted during the evaluation of the calculus/physics program coupled with a review of the literature led me to ask “what is going on” with the manner in which the calculus/physics students use physics to construct conceptualizations of calculus concepts. The present research study attempts to answer the question of “what is going on” by exploring how students draw upon physics concepts to inform their
understanding of calculus concepts. In the next section, I give a brief introduction to the theoretical framework for the present research study. The purpose of the theoretical framework is to help shape the research questions, influence the methodology, and direct the collection of data.

**Overview of the Theoretical Framework**

The theoretical framework developed to support and guide the present study has three main components: (1) The notion of concept image introduced by Tall and Vinner in 1981; (2) A definition of representation developed from a number of such theories discussed in the literature; and (3) Constructivist learning theory.

In order to understand and describe the cognitive aspect of how students are using physics to help them understand calculus concepts, I turned to Tall and Vinner’s (1981) notion of concept image. The idea of concept image has been used by a number of researchers who study mathematics learning (Azcarate, 1991; Stump, 1997; Schwarz, & Hershkowitz, 1999). Basically, Tall and Vinner introduce concept image to describe the mental pictures and all of the processes associated with calling forth the mental picture and extracting information from the mental picture of a concept. Tall and Vinner do not discuss in detail how a student develops a concept image or the cognitive connections made between various concept images. Rather, they use concept image as a way of talking about students’ mental pictures without becoming immersed in cognitive science theory about mental models and mental processing.

Since I am concerned with re-constructing students’ concept images, I am concerned with their mental activities. Specifically, I am looking to try to answer the
question, “What does a student’s concept image of average rate of change/derivative/integral look like?”

The second piece of my theoretical framework provides a rationale for my definition of ‘representation’ and guides the subsequent use of the word throughout this study. One of the most prominent difficulties with the use of the word ‘representation’ is the dualistic nature of its interpretation: a representation could define an internal image or an external icon. Several theories have developed over the past few decades that address the problem of the internal/external dichotomy. These theories seem to fall into one of the four following categories:

1. Internal representations, although distinct from external representations, depict ontological reality (Putnam, 1988). Implicit in this theory is the assumption of a reality external to the individual. A representation is a depiction of something external to the individual.

2. The processes of mathematical thinking occur through interplay between external representations and internal mental processes, including internal representations (De Corte, Greer, & Verschaffel, 1996). Researchers who work under this assumption generally seek to build theoretical models of students’ mental processes through observation of their behavior and build theories of translation between internal and external representations (Kaput 1987a; Goldin 1987; Goldin, 1992).

3. Internal images are presentations of the individual’s constructed reality. It makes no sense to talk about bridging a gap between internal and external representations because that gap does not exist given the underlying epistemology (von Glasersfeld, 1987a; 1995).
4. The discussions of internal and external representation are rejected because they ignore the possibility that a representation can be both internal and external or neither internal nor external (Nemirovsky & Noble, 1997).

My hypothesis is that students develop a concept image of a mathematical concept through experiences with that concept. Students then use conventional mathematical contexts (e.g. graphs, symbols) to express their thinking (concept images) much as we use a conventional language to express our thoughts. When students use conventional mathematical contexts to communicate their presentations of their mental images, these contexts become representations of their mental images. My claim is that students who have a rich understanding of a concept have facility using multiple representations to talk about that concept.

Finally, the third piece of my framework, which guides my thinking about students’ learning, is constructivist learning theory. Specifically, Piaget’s scheme theory describes how students acquire new knowledge. Piaget’s scheme theory or action scheme theory asserts that students, when presented with a learning situation, may recognize aspects of the situation as matching past experience and proceed in acting within the situation as dictated by past experience. If the students’ actions yield an unexpected result, then the student will revisit the situation and modify his/her behavior or thinking to accept the new result. Piaget defines this process as the learning process.

While Piaget’s scheme theory accounts for the individual student’s learning process, critics have claimed that Piaget ignores important social and environmental factors that are crucial to the learning process (Phillips & Soltis, 1991). Social constructivists maintain that humans are social beings and in turn, learning is a social
activity. Social constructivists claim that students are always picking up cues about knowledge from teachers, parents, and other students. Thus, there is a myriad of environmental influences that shape what and how students learn. Since the context of the calculus/physics course is central to this study, I cannot ignore the social and environmental factors that influence students' learning, thus I also draw upon theories of social constructivism to guide my work, especially during the participant-observation phases of data collection.

**Statement of Purpose and Goals**

The purpose of this study is to investigate students' learning about and understanding of calculus concepts. In particular, I am interested in the conceptualizations of average rate of change, derivative, and integral developed by first-year, college-level calculus students in the context of an interdisciplinary calculus/physics course. I am specifically investigating average rate of change, derivative, and integral because these concepts are central to the study of calculus and were a major focus of the interdisciplinary calculus/physics course. The following research goals are guided by my theoretical framework and framed within the context of the interdisciplinary calculus/physics course:

1. To investigate the manner in which students use physics to aid in their conceptualization of calculus concepts. In particular, to develop a classification scheme for the way students use physics to help them understand calculus concepts.
2. To explore of students’ conceptualization of calculus concepts. Particularly, I will consider the results of this study in light of the results of similar investigations reported in the literature.

3. To describe students’ concept images of average rate of change, derivative, and integral.

4. To synthesize the mathematics and physics education literatures. Specifically, to identify areas where the physics education literature supports findings from the mathematics education literature.

The goals listed above helped to narrow the focus of the present investigation. The research goals, along with the theoretical framework and review of the literature shaped the research questions. Specifically, two key observations resulted from my review of the literature: The mathematics and physics education research to date largely had been conducted in isolation from one another. However, both the mathematics and physics education literatures yielded similar results with respect to student understanding of certain concepts. These two observations led me to conclude that the mathematics education literature could be extended by examining students’ understanding of calculus concepts within a specific context, namely within an integrated calculus/physics program. The research questions that developed out of my review of the mathematics and physics education literatures and my work as an evaluator of the Calculus/Physics program at the University of New Hampshire are presented in the next section.
Research Questions

The present research study examined students’ understanding of calculus concepts. The following major question was addressed in this investigation:

How do students draw upon physics concepts to inform their understanding of rate of change, derivative, and integral?

Additionally, the following sub-questions were investigated:

1. Do students’ misunderstandings of fundamental physics concepts misinform their understanding of calculus concepts?
2. Do students consistently use physics in a certain way to help them understand calculus concepts?
3. How do students view the relationship between derivative and integral?

Summary

The present study has been influenced by work in the area of representation theory, constructivist learning theory, and the notion of concept image. The present study attempts to explore students’ conceptualizations of calculus concepts through the lens of their experiences working with physics concepts. The next chapter explains the pieces of the theoretical framework in more detail and highlights the way that the pieces of the framework fit together to guide the present study.
CHAPTER II

THEORETICAL FRAMEWORK

Introduction

In order to make sense of how students come to conceptualize calculus concepts, it is important to consider both how the students experience the concepts and how the students mentally organize information about the concepts. I have drawn on three theoretical perspectives to guide this study: Piagetian and social constructivist theories of learning, a definition of representation, and the notion of concept image set forth by David Tall and Shlomo Vinner (1981).

The constructivist learning theory serves to ground the learning of students in their past and present experiences. In addition the students' reflection upon their experiences is an important component of learning. Certain assumptions follow from assuming a constructivist perspective: Students use their experiences to make sense of problems and contexts of problems; all knowledge is constructed; cognitive structures, which are activated in the process of construction, are continually modified by the learner. Furthermore, the constructivist perspective also informs the methodology employed by researchers who assume a constructivist theory of learning. Typically, qualitative methodologies are employed in such studies in order to provide rich descriptions of the complex learning process (Noddings, 1990; Ernest, 1998).
Research on the development of concept image informs the identification of students' uses of multiple representations of calculus concepts and the connections between representations. Tall and Vinner's concept image ideas were appealing to me for a number of reasons: (1) Tall and Vinner's definition of concept image is broad enough to encompass a multitude of mental structures and processes; (2) Many investigators who study students' conceptions or understanding of undergraduate mathematical concepts draw upon the notion of concept image to frame their work (Schwarz, B. B. & Hershkowitz, R., 1999; Stump, 1997; Azcarate, 1991); and (3) The notion of concept image links up with the constructivist perspective in my framework.

The role of representation in my framework serves to define more clearly what is meant by concept image and acts as a link between the two main parts of my framework: constructivist learning and concept image. Von Glasersfeld (1987b) warns that, "when it [the word representation] is used in technical contexts but without a specific definition, it tends to remain opaque" (pp. 215). As I attempted to define what I meant by 'representation' I stumbled upon my own opaque usage of the word. At first, I was using the word to describe both the mental pictures in the mind of the student and conventional mathematical tools, such as symbols and graphs, used by the mathematical community at large to describe mathematical phenomena. What was not clear to me was how to justify using the same word, 'representation' to describe both situations. The theory of representation presented in this work portrays my current thinking about defining 'representation' and how my definition of representation complements the constructivist learning theory and concept image theory.
My framework has evolved since the inception of this study because my own views and opinions matured as I considered a variety of authors’ ideas and positions. The development of my framework can be compared to the focusing of a lens on a camera. When I set out to express the theories and concepts guiding this study, my lens was wide and unfocused which caused the details of my framework to appear blurred. My lens became more focused as I attempted to synthesize various theories and identify my own beliefs and assumptions. The focusing of my lens has allowed me to identify and discuss the details of my framework that I will present in this chapter.

The discussion of the theoretical framework begins with an overview of Tall and Vinner’s notion of concept image and concept definition. Next, I will talk about the definition of ‘representation’ I used in the present study and compare four different theories of representation. Finally I discuss how the notion of concept image fits into a constructivist theory of learning.

**Concept Image**

In order to understand and describe the cognitive aspects of how students are using physics to help them understand calculus concepts, I needed a way to talk about what the students’ cognitive structures looked like. Tall and Vinner’s (1981) notion of concept image allowed me to focus on how the students are working with multiple representations of calculus concepts without getting caught up in the details of cognitive description often found in other theories of cognition (see De Corte, Greer, & Verschaffel, 1996 for a description of some general theories of cognition.).
Additionally, Tall and Vinner admit that students in an informal setting (everyday life) have experienced many mathematical concepts introduced in the classroom and so students already have a mental picture of many mathematical concepts when they enter the classroom. Tall and Vinner (1981) claim that,

Many concepts that we meet in mathematics have been encountered in some form or other before they are formally defined and a complex cognitive structure exists in the mind of every individual, yielding a variety of personal mental images when a concept is evoked (pp. 151).

Specifically students enter a calculus class having some experience with certain calculus topics. For instance, a student may not have seen a formal definition of rate of change, but he or she has experienced the phenomena of velocity while traveling in a car. Thus the student previously encountered the concept of rate of change without having the term ‘rate of change’ formally defined.

Tall and Vinner (1981) use the term *concept image* to discuss the mental pictures that students have of concepts. Tall and Vinner (1981) define *concept image* as that which “describes the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes” (pp. 152). For example, a child might develop a concept image of ‘dog’ based on his/her experiences and encounters with dogs. If the child only encounters large dogs, such as Golden Retrievers, then the child’s concept image of ‘dog’ may include a picture of a large, furry creature, with four legs and a long, wagging tail that makes a ‘woof’ sound. This concept image may cause the child problems in the future when he/she meets a bulldog. The bulldog may conflict with the child’s concept image of dog being large and furry with a long, wagging tail.
Tall and Vinner (1981) first used the notion of concept image to explore students' understanding of limits and continuity. Tall and Vinner found that students' concept images of limit and continuity often conflicted with the formal, mathematical definitions of limit and continuity. Tall and Vinner (1981) elicited information about students' concept images through the administration of questionnaires. Tall and Vinner used the students' responses to the questionnaires to develop general descriptions of students' concept images. For example, some students held a concept image of continuity that involved a graph having no gaps or holes (Tall & Vinner, 1981, pp. 167).

Since Tall and Vinner introduced the notion in 1981, other researchers have used the idea of concept image in a variety of ways. For instance, Schwarz and Hershkowitz (1999) investigated the role of prototypical examples in students' concept images of function. Schwarz and Hershkowitz studied students enrolled in a Grade 9 program based on three cycles of curricula. Each cycle is based on the previous one, but extends beyond the scope of the previous cycle. The cycles were designed to build upon the students' already existing concept images to develop more sophisticated concept images in the students. Schwarz and Hershkowitz characterized the students' concept images during the second and third cycles and then compared and contrasted the characterizations for each student.

Stump (1997) investigated pre-service and in-service secondary mathematics teachers' understanding of various representations of slope and their knowledge for teaching the concept of slope. Stump designed interview and survey questions to probe the teachers' concept images of slope. Stump looked for patterns among the teachers' concept images of slope and reported on her findings.
Slavit (1994) studied the development of high school Algebra II students' concept images of function. Slavit studied the development of the students' concept images as part of a larger study aimed at investigating the effect of graphing calculators on students' conceptions of function. Slavit used the notion of concept image to discuss students' translation ability between functional representations.

Tall and Vinner's (1981) notion of concept image has been widely used by mathematics education researchers, especially those investigating student understanding of tertiary mathematics. The notion of concept image is a tool that allows me to discuss students' mental pictures and processes of various calculus concepts. The concept image notion helped me organize my interpretation of the students' conceptualizations of average rate of change, derivative, and integral. The use of concept image in the data analysis will be shown in Chapter V.

**Representation**

The idea that symbols represent information is central to any definition of mathematics. However, closer scrutiny of what it means for symbols to 'represent' has led mathematics education researchers to examine the question: *Representation of what, for what purpose?* (Vergnaud, 1987) and more fundamentally, *What does it mean to represent?* The complexity of the task of attempting to answer the above questions can be attributed, in part, to the myriad of meanings of the word 'representation'.

The word 'representation' takes on a variety of different meanings in the English language. The Oxford English Dictionary (2001) gives a number of distinct definitions:

b. Appearance; impression on the sight. Obs.

2. a. An image, likeness, or reproduction in some manner of a thing.
b. A material image or figure; a reproduction in some material or tangible form; in later use esp. a drawing or painting (of a person or thing).
c. The action or fact of exhibiting in some visible image or form.
d. The fact of expressing or denoting by means of a figure or symbol; symbolic action or exhibition. Also pl.
e. Math. The image of a homomorphism from a given (abstract) group to a group or other structure having some further meaning or significance; such a homomorphism.

3. a. The exhibition of character and action upon the stage; the (or a) performance of a play.
b. Acting, simulation, pretence. rare.

4. a. The action of placing a fact, etc., before another or others by means of discourse; a statement or account, esp. one intended to convey a particular view or impression of a matter in order to influence opinion or action.
b. Insurance. A special statement of facts relating to the risk involved, made by the insuring party to the insurer or underwriter before the subscription of the policy.

5. a. A formal and serious statement of facts, reasons, or arguments, made with a view to effecting some change, preventing some action, etc.; hence, a remonstrance, protest, expostulation.
b. Sc. Law. 'The written pleadings formerly presented to a lord ordinary in the Court of Session, when his judgment was brought under review' (Bell).

6. a. The action of presenting to the mind or imagination; an image thus presented; a clearly-conceived idea or concept.
b. The operation of the mind in forming a clear image or concept; the faculty of doing this.

7. a. The fact of standing for, or in place of, some other thing or person, esp. with a right or authority to act on their account; substitution of one thing or person for another.
b. Law. The assumption by an heir of the position, rights, and obligations of his predecessor. right of representation, the right whereby the son of an elder son deceased succeeds to his grandfather in preference to the latter's immediate issue (see also quot. 1838).

8. a. The fact of representing or being represented in a legislative or deliberative assembly, spec. in Parliament; the position, principle, or system implied by this.
b. The aggregate of those who thus represent the elective body.

In everyday language, as seen in the above definitions, the word representation most commonly refers to the act of someone or something standing for another person or
object. In this common usage of the word, there seems to be an underlying assumption that the represented person or object is the ‘true’ target of discussion and the representation is merely a copy or a stand in. What is important here is that the focus is not on the representation, itself, but on what the representation stands for; the ‘true’ object. This common usage of the word representation in our language is the source of some of the controversies and discussions surrounding the use of the word ‘representation’ in the mathematics education literature.

For example, the assumption that a represented person or object is a ‘true’ target of discussion and the representation is a stand-in can be translated to describe learning as a process by which students construct mental representations that mirror external constructs. In this example, the external constructs are the ‘true’ targets of discussion and the students’ mental representations are copies of that ‘true’ external target. In such a learning situation, students are presented with instructional materials which make it possible for them to construct correct internal representations of mathematical knowledge (Cobb, Yackel, & Wood, 1992). Holding such a view of learning gives rise to what Berieter (1985) calls the learning paradox: Learners must grasp concepts or procedures more complex than those they already have available for application.

Overcoming the learning paradox has been a challenge for all educators. In particular, mathematics educators have struggled with the dual use of the word representation: Internal representations that describe mental images and pictures and external representations that are made up of symbols, graphs, and physical objects.
Several theories have developed over the past few decades that address the problem of the internal/external dichotomy. These theories seem to fall into one of the four following categories:

1. **Representational View of Mind**: Internal representations while distinct from external representations depict ontological reality. This view is referred to as the representational view of mind (Putnam, 1988).

2. **Translation**: The processes of mathematical thinking occur through interplay between external representations and internal mental processes, including internal representations (De Corte, Greer, & Verschaffel, 1996). Researchers who work under this assumption generally seek to build theoretical models to describe students’ observable behavior (Kaput, 1987a; Goldin 1987; Goldin 1992). This theory also involves discussions about translating between internal and external representations.

3. **Re-presentation**: Internal images are presentations of the individual’s constructed reality. It makes no sense to talk about bridging a gap between internal and external representations because that gap does not exist given the underlying epistemology (Mason, 1987b; von Glasersfeld, 1987b). I will refer to this viewpoint as the representational theory.

4. **Lived-in-Space and Transitional Tools**: All discussions of internal and external representation are rejected because they ignore the possibility that a representation can be both internal and external or neither internal nor external. I will refer to this stance as the lived-in-space theory due to Nemirovsky & Noble (1997) who presented this argument and the lived-in-space solution to the internal/external dichotomy problem.
My presentation and discussion of the theories of representation serves two purposes: (1) to give the reader an historical perspective on the development of different theories of representation in mathematics education. (2) To compare and contrast these theories with each other as to give rise to my interpretation of representation.

<table>
<thead>
<tr>
<th>THEORY</th>
<th>WHO</th>
<th>WHAT THEY SAY</th>
<th>IMPLICATIONS FOR INSTRUCTION</th>
</tr>
</thead>
<tbody>
<tr>
<td>Representational View of Mind</td>
<td>Putnam (1988)</td>
<td>Internal representations model an external ‘real’ world.</td>
<td>The goal of instruction is for students to build their internal representations in order to mirror external representations.</td>
</tr>
<tr>
<td>Translation</td>
<td>Goldin (1987, 1992) Kaput (1987)</td>
<td>Internal and external representations are different but related.</td>
<td>The goal of instruction is for students to become fluent in using different models (for example, symbols and pictures.) to present their world.</td>
</tr>
<tr>
<td>Re-presentation</td>
<td>Von Glasersfeld (1987, 1995) Mason (1987)</td>
<td>Internal imagery is taken as primary, that is, mental constructs are the primary basis from which students build their mathematical knowledge.</td>
<td>The goal of instruction is for students to become fluent in using different models (for example, symbols and pictures) to present their world.</td>
</tr>
<tr>
<td>Lived-in-Space and Transitional Tools</td>
<td>Nemirovsky &amp; Noble (1997)</td>
<td>The internal/external dichotomy is rejected because it ignores the cases that a visualization can be both internal and external or neither internal nor external.</td>
<td>The goal of instruction is for students to use transitional objects to help mathematical ideas and symbols become part of their lived-in space.</td>
</tr>
</tbody>
</table>

Table 1: Four Major Representation Theories
The chart above outlines the four major theories of representation, those who have written about them, and each theory's implications for instruction. Each theory will be expanded on and I will describe my interpretation of the inadequacies of each theory for the purposes of the present study.

**Representational View of Mind**

The assumption underlying this theory of representation is that internal representations depict ontological reality. Thus whatever a student pictures mentally is an image of an outside world, not a world constructed by the student. The goal of the learner is to match his/her mental construct with an external construct (Putnam, 1988). Rorty, as cited in Cobb, Yackel, and Wood (1992) describes the representational view of mind as follows:

> To know is to represent accurately what is outside the mind; so to understand the possibility and nature of knowledge is to understand the way in which the mind is able to construct such [internal] representations (pp. 3, brackets in Cobb, Yackel, and Wood).

Thus the representational view of mind characterizes learning as a way in which students transform their mental images, or concept images to reflect the structure of external representations.

The representational view of mind, unlike the other three theories of representation, seems to oppose the underlying philosophical assumptions of some constructivist theories of learning. In particular, the representation view of mind assumes the existence of a reality separate from one's own experiences. Thus, in the representation view of mind, teachers and students are not treated as independent creators of their ways of knowing, but rather as reflectors of a world separate from their own actions and interactions (Cobb, Yackel, & Wood, 1992, pp. 15).
Some potential difficulties arise when one attempts to combine tenets of constructivism with the representational view of mind. Cobb, Yackel, and Wood (1992) describe four such theoretical difficulties: (1) A tension between the view of learning as a process in which students actively construct knowledge as they make sense of their worlds and learning as recognizing relationships presented in instructional materials. (2) The semantic theories underlying constructivism and the representational view of mind are incommensurable. (3) In the representational view of mind, the teacher’s expert interpretations are projected into the learner’s environment and treated as mind-independent external representations. (4) The representational view of mind rejects notions that mathematical meanings are socially and culturally situated (Cobb, Yackel, & Wood, 1992, pp. 6—7). Furthermore, Cobb, Yackel, and Wood (1992) argue that the representational view of mind is at odds with reform efforts in mathematics education since the representational view of mind seems to discourage students’ development of conceptual meanings. I agree with Cobb, Yackel, and Wood’s criticism of combining pieces of constructivism with the representational view of mind, as outlined above. For the purposes of the present study, the representational view of mind falls short of providing an adequate theoretical foundation mainly because the dichotomy between internal and external representation as proposed by the representational view of mind is directly opposed to my own beliefs about knowledge, learning, and teaching. I believe that subscribing to a dichotomy between internal and external representation leads to a belief that the goal of teaching and learning is that the students’ internal representation will match some external representation as decided by the teacher and that knowledge is external to the learner, rather than created by the learner.
Additionally, the representational view of mind ignores the social and contextual nature of mathematics learning. The focus of the representational view of mind is to help explain how knowledge gets into students’ heads, not how students create ways of knowing through their interactions with others and their environments. The other three theories of representation that I will discuss embrace the idea that learning mathematics is a social as well as a cognitive process.

**Translation**

The translation theory of representation assumes the existence of both internal and external representations, but unlike the representational view of mind, translation theorists assume that the interplay between internal mental operations and external representations is a complex social and cognitive process. Some researchers attempt to build theoretical models of students’ internal representations by observing students’ behavior as they interact with external representations (for example, Goldin, 1987; 1992). Other researchers attempt to construct models to explain how students bridge the gap between internal and external representations (for example, Kaput 1987b; Janiver, 1987; Lesh, Post & Behr, 1987).

Kaput (1987a, 1985, 1982) has worked with a model that extends Palmer’s theory (1977) that any concept of representation must involve two related but functionally separate entities, namely a domain and a co-domain. Kaput (1987a) outlines five necessary elements of a representation: (1) the represented world, (2) the representing world, (3) what aspects of the represented world are being represented, (4) what aspects of the representing world are doing the representing, and (5) the correspondence between the two worlds (Kaput, 1987a, pp. 23). Following Kaput’s outline, the represented world
is the learner’s world and the representing world is the world of mathematical symbols.

Cobb, Yackel, and Wood (1992) claim that

This line of research rejects the view that mathematical meaning is inherent in external representations and instead proposes as a basic principle that the mathematical meanings given to these representations are the product of students’ interpretive activity (pp. 2).

Along these lines, Confrey (1990) argues that pedagogical devices become representations only when students use these devices to express a conception. Thus, a calculus context becomes a representation only when a student uses that context to express an internal presentation. For example, a student may define a derivative as “the graph of the slopes of the tangent lines at each point of the original function.” This student is using the graphical context to express her internal conception of derivative. The student imposes meaning onto the graphical context and thus the graphical context becomes an external representation of derivative for the student.

However, von Glasersfeld (1995, 1987b, 1984) cautions using the word ‘representation’ to refer to two related but separate entities (in this case, the learner’s mental constructs and external representations of those constructs). He argues that using the word ‘representation’ to refer to both internal and external representations and ‘translation’ to refer to the bridge between internal and external representations is absurd since, “there is no logically possible access to what they are supposed to represent” (von Glasersfeld, 1987b, pp. 224). Von Glasersfeld carves out a theory distinct from the Translation theory of representation that I will refer to as the Re-presentation theory.

I agree with von Glasersfeld’s cautionary statements about using the word ‘representation’ to refer to both internal constructs of the student and external
manifestations of those constructs. Furthermore, I believe that using the single word
'representation' causes confusion for the reader, since it may not be obvious if the use of
the word representation refers to an internal or external image. In terms of the present
study, I needed to make a clear distinction between the students' internal
conceptualizations and the external contexts that the students encountered the problems.

Re-presentation

Von Glasersfeld (1987b, 1995) proposes using this hyphenated spelling of the
word since he maintains that internal or mental 'representations' are really presentations
of the individual's constructed reality and if we use symbols, pictures, or other external
objects to manifest them, then we are re-presenting the internal images. The subtle
distinction between a Re-presentation theory and a Translation theory is that a Re-
presentation theorist subscribes to the belief that mental constructs are the primary basis
from which students build their mathematical knowledge. The internal experiences of a
learner are primary because they are the learner's world (Mason, 1987a, pp. 207).
Translation theorists, on the other hand, believe that the interplay between mental
constructs and external representations is the primary basis from which students build
their knowledge, not simply internal conceptions.

Von Glasersfeld (1987b) claims that the common usage of the word
'representation' implies, as mentioned above, the existence of an object - external to the
learner - that is being represented. Mason (1987a) claims that using the word
'representation' is not a clear way to describe what goes on inside a person's mind
because their inner experiences are their world, not a representation of the world (pp.
207). Von Glasersfeld (1987b) proposes using the word 'presentation' or 'conception' to

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describe the mental constructs of the learner and 're-presentation' to denote the
manifestation of internal conceptions with pictures, symbols, or graphs.

The theory of Re-presentation, while subtly distinct from the theory of
Translation, acknowledges the existence of an internal/external dichotomy problem. In
fact, the three theories discussed above all work from the assumption that an
internal/external dichotomy problem exists. I was uncomfortable with the assumption that
an internal/external dichotomy exists because I felt that von Glasersfeld had not fully
considered the cases that an image may reside neither inside or outside a person’s mind
or that an image may reside both inside and outside a person’s mind. Furthermore, the
assumption that an internal/external dichotomy problem exists creates the need for the
researcher to ascertain whether a visual image is internal to the student or external, such
as on a piece of paper. The next theory that I discuss diverges from the previous three in
that it rejects the standard internal/external dichotomy problem in favor of a theory that
allows for the possibilities of an object or picture to reside both or neither inside and
outside the mind of the learner.

Lived-in-space and Transitional Tools

Nemirovsky and Noble (1997) put forth an alternative solution to the
internal/external dichotomy problem by formulating a psychological perspective that will
help us analyze an individual’s constructive activity by challenging the convention that
any given object or picture must reside either inside or outside a person’s mind.
Nemirovsky and Noble reject the internal/external dichotomy on the basis that it does not
leave room for the possibility of an object or picture residing neither inside or outside a
person’s mind or residing both inside and outside a person’s mind at the same time.
"When researchers attempt to pin down what learners’ processes of visualization are, a common difficulty arises from the need to describe whether the visual image is 'in the student’s mind' or ‘outside’ the student, on a piece of paper or a computer screen” (pp. 101). Nemirovsky and Noble chose to study the development of mathematical visualization in terms of a learner’s experience with lived-in spaces and transitional objects.

Nemirovsky and Noble (1997) contend that a lived-in-space is “not ‘carried’ by the individual, but created in an ongoing process that involves memories, intentions, and the situation at hand” (pp. 105). I am interpreting their idea of lived-in-space as a combination of lived experiences (which includes, for example, memories and mental pictures.) and an evolutionary process of the mental presentations. The evolutionary process of mental presentations refers to the notion that the more individual interacts with a concept, the individual’s mental picture of that object becomes more detailed. This stance is meant to account for the differences between a person recalling an event, moment, or object and a person actually living through the event or moment or interacting with the object. Nemirovsky and Noble contend that the most fundamental quality of whatever inhabits a lived-in-space is its property of being both internal and external. For example, a student who talks about the motion of a hypothetical cart on a track to help him/her conceptualize properties of derivative is using the cart and track as tools which reside both internally (the student imagines the motion of the cart on the track) and externally (the physical existence of the cart and track). The cart and track fall into a classification that Nemirovsky and Noble (1997) call transitional tools.
Nemirovsky and Noble introduce the concept of transitional tools into their theory as an example of why it is necessary for them to consider the possibility of representations as both internal and external. The idea of transitional tools has been adopted from Winnicott’s (1971) ideas of ‘transitional phenomenon’. Transitional objects are objects in the environment that both separate a learner from another physical object and bring him/her closer to it using symbols, graphs, or other mathematical contexts (Nemirovsky & Noble, 1997, pp. 123). When using transitional objects, there is a tendency to anthropomorphize them. Nemirovsky and Noble claim that, “…enlivening tools with human qualities is a pervasive response that most of us resort to as we strive to grapple with new tools and new situations” (pp. 124). The authors claim that anthropomorphizing is one way that an individual makes transitional objects part of his/her lived-in space. Another way that an object becomes part of a lived-in space is through remembering. An object that has sentimental value often becomes more than an object, but a trigger of certain memories. Nemirovsky and Noble (1997) claim that a sentimental object becomes part of one’s lived in space as “the boundaries between the object as a thing and the object as [the memories] get dissolved” (pp. 125).

Nemirovsky and Noble also propose an idea that uses experience as a way to make generalizations (versus mathematical generalization). Here a generalization refers to forming general notions obtained from the observation and comparison of individual facts or appearances, while a mathematical generalization denotes the formal process of symbolically extending a specific result to a larger class of objects.

For example, Karen, a student in Nemirovsky & Noble’s study, connected memories of negative velocity graphs and counting negative numbers.
When we did our velocity graphs [in a physics class at school]...when you’re going towards the [motion] sensor it was negative. But when you went away it was positive. Even if you were still increasing speed it [the velocity graph] would go the other way [down]. And it’s like counting negatives, the numbers go up, and it gives you the illusion of it getting bigger really when it isn’t...(pp. 116, bracketed text in original).

These two mathematical phenomena - negative velocity graphs and counting negative numbers are not connected by any general mathematical principle, rather they are connected by Karen’s general surprise that something can get bigger as it is decreasing. Karen describes this phenomenon of something getting bigger as it decreases as an ‘illusion’. Notice that Karen’s moment of remembering was not a mere retrieval of information, but a reawakening of feelings of past experiences. Karen’s feeling of surprise at the behavior of the velocity graph was similar to her feelings of surprise when she initially worked with negative numbers.

Nemirovsky and Noble’s ideas about lived-in-space and transitional tools are very appealing as descriptors of the role of environmental objects in students’ understandings of mathematical concepts. However their theory falls short of addressing the need for vocabulary to define students’ mental images and their manifestations. Thus, I needed to combine aspects of Nemirovsky and Noble’s theory with von Glasersfeld’s focus on vocabulary as I developed my working definition of representation.

Nemirovsky and Noble (1997) contend that “The most fundamental quality of whatever dwells in a lived-in space is its being at once internal and external, its being ‘me and not-me’” (pp. 125). In the next section, I rely on this fundamental property of objects in an individual’s lived-in space to formulate my own theory of representation.
My Working Definition of Representation

I have presented a comparison of four theories of representation prevalent in the mathematics education literature. I compared and contrasted these four theories in order to give an historical perspective on the development of representation theories and to set the stage for the discussion of my own theory of representation. I will now lay out my definition of representation, which draws heavily upon notions set forth by Nemirovsky and Noble (1997) and heeds the warnings of Von Glasersfeld (1987b) who advises that careful attention be paid to the meanings and definitions of words.

Von Glasersfeld (1987b) suggests that a definition of ‘representation’ precede the formulation of a theory of representation. “It would seem indispensable that, at the outset, we clarify as best we can what kind of conceptual structure we have in mind when we say ‘representation’” (von Glasersfeld, 1987b, pp. 215). I will use the word ‘representation’ to mean a mathematical context that is used by a student to express a conception, in the flavor of Confrey (1990). The students’ mental pictures I will refer to as “presentations” or “conceptions”. In particular reference to the present study, a student’s concept image consists of, among other things, presentations of the mathematical representations of a concept. For example, a student may have a conception of the derivative of $x^2$ as a straight line going through the origin on a Cartesian plane. If the student sketches a picture of $2x$ on an x versus y graph, the students’ picture is a representation of his/her conception of the derivative of $x^2$.

I believe that individuals use transitional objects (such as the carts and tracks) to help mathematical ideas and symbols become part of their lived-in space. Anthropomorphizing and remembering are two ways that mathematical ideas and
symbols become part of an individual’s lived-in space. A fundamental feature of those objects residing in lived-in space is their property of being both internal and external to the individual (Nemirovsky & Noble, 1997, pp. 125). As an individual repeatedly works with transitional objects and frequently encounters mathematical ideas and symbols through the use of transitional tools, the individual develops a stronger concept image. Individuals then use conventional mathematical contexts (e.g. graphs, symbols) to express their concept images, similar to the use of a conventional language to express thoughts. When an individual uses conventional mathematical contexts to communicate his/her presentation of his/her mental images, these contexts become representations of their mental images. Thus, the images in a student’s mind I will refer to as ‘presentations’. Furthermore, I will refer to the external medium that the student uses to express his/her presentations as a representation.

My interpretation of representation, drawn from various sources, helps me to organize and describe the results for each student. Problems were presented to the students in one context but the students sometimes referred to other contexts as they solved the problems. For example, many students converted data from a table into a graph and solved problems originally posed in a numeric context in a graphical context. In this case, the student rejected the numeric context of the problem in favor of another context (graphical) to communicate his/her mental images of the problem. The graphical context, introduced to the problem by the student, is a representation.

Summary

I began this section by listing the various definitions of the word ‘representation’ from the Oxford English Dictionary. It would seem that a person who has a rich
conception of the word ‘representation’ would be able to articulate many of the definitions listed above. This person’s mental picture of ‘representation’ would be manifested by his or her language. If mental images and thoughts are manifested by language, then, similarly, certain mental images and thoughts are manifested by mathematical symbols and syntaxes (namely symbolic, numeric, graphical, physical). If we can describe an object in a number of different ways using our language, we would say that we have a well-developed conception of that object (see example of the definitions of ‘representation’ above). Similarly, if we can describe a mathematical idea in various syntaxes, we have a well-developed conception of that mathematical concept. Standard mathematical syntaxes become representations as the learner uses these syntaxes to express a conception.

The previous two sections discuss the internal structures of the learner. I have drawn on Tall and Vinner’s (1981) notion of concept image as a place to point my lens. My work with Representation Theory has helped me focus my lens on a specific component of concept image, namely internal representation. However, in order to complete the picture, it is necessary to discuss how the lens works, that is how the student creates a concept image. The next piece of my framework discusses a constructivist theory of learning.

Constructivist Theories of Knowing and Learning

The work of Jean Piaget has had a profound impact on mathematics teaching, learning, and mathematics education research. Piaget’s work has been accepted with enthusiasm by some, criticized by others, and interpreted and adapted by many. One of
the main learning theories to grow out of Piaget’s work is the constructivist learning
theory. Von Glasersfeld (1995) describes Piaget’s learning theory in the following way:

The learning theory that emerges from Piaget’s work can be
summarized by saying that cognitive change and learning in a
specific direction take place when a scheme, instead of producing
the expected result, leads to perturbation, and perturbation, in turn,
to an accommodation that maintains or re-establishes equilibrium
(pp. 68).

Let me first note that implicit in von Glasersfeld’s statement is an underlying assumption
that the learner must be active in order for cognitive change to take place. This implicit
assumption should become more apparent as I define the terms in von Glasersfeld’s
definition. Additionally, Piaget’s work is scattered with references to his view that
knowledge arises from activity and engagement with the environment (Piaget, 1970a;
1970b; Piaget & Inhelder, 1971). Piaget and Inhelder (1971) use a metaphor of finding
one’s way in an unfamiliar town to describe the learning process:

We all know that we discover and remember the lay-out of a strange town
much better if we walk about in it on our own, and remain responsible for
our own wrong turnings, rather than rely on a friend to show the way,
although the perceptual data is comparable in both cases (pp. 229)

Piaget’s Theory of Knowledge

Much of Piaget’s theory of knowing and learning is derived from his work as a
biologist. For example, Piaget’s notion of ‘action scheme’ is derived explicitly from the
biological idea of ‘reflex’ (von Glasersfeld, 1995, pp. 56). This is an important point,
because von Glasersfeld, in his interpretation of Piaget’s theory, often looks toward
biology for examples of Piaget’s conceptions.

A theory of knowledge must presuppose any theory of learning and Piaget’s
theory of knowledge is one of the most controversial and misinterpreted. Again, drawing
on his training as a biologist, Piaget viewed his theory of knowledge more aligned with Darwin’s theory of evolution than other more traditional theories of knowledge at that time (c.f. Locke, Skinner). Instead of viewing knowledge as representations of an ontological reality, Piaget, drawing from Darwin’s theory of evolution and Kant’s transcendental philosophy, theorized that knowledge is the result of a construction of structures grounded in experience. Then, instead of viewing cognition as the producer of representations of an ontological reality, Piaget viewed cognition as an instrument of adaptation (Von Glasersfeld, 1995, pp. 59).

**Piaget’s Action Scheme Theory**

Von Glasersfeld’s (1995) interpretation of Piaget’s scheme theory has been widely cited throughout the mathematics and science education literature (e.g. Wessel, 1999; Hardy & Taylor, 1997; Confrey, 1994; Ernest, 1994). I chose to work with von Glasersfeld’s interpretation since it seems to be the most reasonable fit with my conception of Piaget’s theory. Also, von Glasersfeld explicitly links Piaget’s ideas to biological notions, an important influence on Piaget’s work.

Piaget’s scheme theory or action scheme theory has its roots in the biological idea of reflex. Piaget thought of the idea of reflex in three parts: a perceived situation, an activity associated with the situation, and a response or result (von Glasersfeld, 1995, pp. 64). Piaget’s conception of reflex in three parts differed from the standard biological interpretation of reflex as a two-part process: stimulus and response. Piaget adopted his model of the reflex to explain cognitive action; this is the root of his action scheme theory and it is outlined in Figure 1.
In Part 1, a situation is presented to the learner. A scheme begins if the learner recognizes aspects of the situation that he/she has experienced in the past. In Part 2, the learner attempts to carry out a specific activity associated with the situation. It is at this point that Piaget's idea of *assimilation* surfaces.

Von Glasersfeld interprets assimilation to signify, "...treating new material as an instance of something known" (pp. 62, italics in original). This is not to be mistaken for an individual incorporating elements of the environment into his/her existing structure. It is important to note that assimilation should be thought of more as a matching of a new situation with prior experience. In this process of matching a new situation with prior experiences, certain aspects of the new situation will be ignored: those aspects that do not match an individual's prior experience. This happens because the individual simply may not perceive those aspects of the new situation that do not match with prior experience. In von Glasersfeld's words,

"The cognitive organism perceives (assimilates) only what it can fit into the structures it already has...it remains unaware of, or disregards, whatever does not fit into the conceptual structure it possesses...In short, assimilation always reduces new experiences to already existing...conceptual structures, and this inevitably raises the question why and how learning should ever take place" (pp. 63).
In order to address von Glasersfeld’s concern about learning, we look back at Part 2 of the action scheme. When the individual attempts to assimilate a new situation he/she has an expectation that will produce a result, shown in Part 3.

If the individual does not achieve an expected outcome, a perturbation results. A perturbation, von Glasersfeld’s translation of the French term that Piaget (1974) originally used to describe the discrepancy between an individual’s expected outcome and the actual outcome, has been translated as ‘disequilibration’ by others (Geber, 1977). I will use the word ‘perturbation’ in this discussion since I am adopting von Glasersfeld’s interpretation of Piaget’s theory. When a perturbation results, it is likely that the individual will review Part 1, the perceived situation. It is in this reviewing process that the individual might notice some of the characteristics of the situation that he/she previously ignored. Two cases arise, as described by von Glasersfeld:

If the unexpected outcome of the activity was disappointing, one or more of the newly noticed characteristics may effect a change in the recognition pattern and thus in the conditions that will trigger the activity in the future. Alternatively, if the unexpected outcome was pleasant or interesting, a new recognition pattern may be formed to include the new characteristic, and this will constitute a new scheme. In both cases, there would be an act of learning and we would speak of ‘accommodation’ (pp. 65-66).

It is important to note that in von Glasersfeld’s interpretation of Piaget’s scheme theory, accommodation takes place only when a scheme does not yield an expected result.

The final concept central to Piaget’s learning theory is that of equilibrium. Equilibrium takes place when a perturbation is eliminated. The elimination of perturbations happens through the process of accommodation. Thus, to interpret von Glasersfeld’s statement of Piaget’s learning theory: Learning occurs when, through a process of accommodation, perturbations are eliminated and equilibrium is restored.
Social Concerns

While Piaget’s work laid the foundation for much of the constructivist theory of learning, he ignored an aspect of learning that some consider critical: the social aspect of learning. Piaget conducted his research on children working in isolation; he was concerned with understanding the cognitive structures the children were building, not the impact of such things as history and culture on learning. Thus, a new branch of constructivism grew out of criticism of Piaget’s original work: social constructivism. Social constructivists maintain that humans are social beings and in turn, learning is a social activity. Social constructivists claim that students are always picking up cues about knowledge from teachers, parents, and other students. Thus, there is a myriad of environmental influences that shape what and how students learn. In fact, Phillips and Soltis state, “Any account of learning that gives short shrift to these diverse social factors must be deficient to some degree” (Phillips & Soltis, 1991, pp. 51).

The present study is guided by the belief that knowledge is constructed through a process of experience and reflective abstraction (see Noddings, 1990, pp. 10). Confrey (1990) claims that,

In mathematics the reflective process, wherein a construct becomes the object of scrutiny itself, is essential. This is not because, as so many people claim, mathematics is removed from everyday experience. It is because mathematics is not built from sensory data but from human activity...As a result, to create such a language we must reflect on that activity, learning to carry it out in our imaginations and to name and represent it in symbols and images (pp. 109).

The notion that mathematics is built from human activity is an essential piece of my framework. Students’ day-to-day experiences are situated culturally and often occur while interacting with other persons. Within these culturally situated interactions with others, students informally experience calculus ideas. Students experience the physical
aspect of some calculus concepts, such as rate of change, before they encounter the formal definition of rate of change in the classroom. Similarly, students learn about physical contexts of derivative and integral when they are exposed to physics ideas of kinematics and force. Students experience such concepts as velocity, acceleration, impulse, and balance, and bring those experiences with them to the calculus classroom.

However, some students encounter misleading prior physics experiences. For example, it has been well documented in the physics education literature that students possess naïve (mis)conceptions about motion (Champagne, Klopfer, & Anderson, 1980; Clement, 1982). For example, students tend to believe that a constant force produces a constant velocity and in the absence of forces, objects are either at rest or slowing down. Furthermore, some students maintain these Aristotelian views about motion even after a year of physics instruction (McCloskey, Caramazza, & Green, 1980; Halloun & Hestenes, 1985a). While previous research has acknowledged the role of real-world experiences contributing to the existence naïve conceptualizations of students' physics conceptions, prior research has not addressed how, if at all, these naïve physics conceptions influences students' learning of mathematics concepts. Recall that one of the goals of the present study is to determine if students' physics misconceptions influences their learning of calculus concepts. I will address this topic in more detail in Chapter III.

The Interplay between the Theoretical Framework and Research Methodology

Noddings (1990) noted that “Acknowledgment of constructivism as a cognitive position leads to the adoption of methodological constructivism” (pp. 10). Methodological constructivism, which primarily involves qualitative research methods, is concerned with describing an individual’s perceptions, thoughts, and intentions in order
to describe the individual’s behavior (Noddings, 1990, pp. 14). The influence of a constructivist theory of learning on choice of research methodology and data analysis has been discussed in a number of recent articles (Greeno, Collins, & Resnick, 1996; Ernest, 1998; Clement, 2000).

The theoretical framework of this study influenced the choice of research methodology in a number of ways: (1) The assumption that all knowledge is constructed and the view that an individual’s knowledge can never mirror an external world led to the adoption of qualitative research methods, which are sensitive to the human features and limitations of knowing. (2) The case study design was chosen as the primary research method since the case study allows for an in-depth investigation of an individual or group of individual’s process for making sense of mathematical concepts. The case study methodology allows the researcher to explore the complexities of the role of prior experience in the learning process. (3) The assumption that learning is a complex, cognitive process led to the choice of clinical interviews to probe students’ thinking and reasoning processes. (4) The assumption that learning is a social process led to the choice of classroom participant-observation to witness the students in their natural learning environment.

Furthermore, an immediate consequence of adopting the constructivist perspective is that all knowledge is perceived as individualized. A research methodology should reflect the constructivist epistemology, giving ample consideration to the individual’s construction of knowledge. The case study methodology allows for the consideration of an individual’s experiences and knowledge constructions. The case
study methodology "convey[s] to the reader what experience itself would convey" (Stake, 1995, pp. 39).

**Research Assumptions**

This research study is supported and informed by several theories and perspectives. Foremost is the assumption that knowledge is a construct of the individual and constructs are formed in response to experiences and active engagement with elements of the environment. Additionally, individuals do not develop understandings in isolation from culture, history, and other persons. Therefore, an individual’s interpretation of his/her experiences is influenced by other persons, the culture of the classroom, and society at large (Jaworski, 1994; Ernest, 1994; 1993; Phillips & Soltis, 1991).

The present study focuses on students’ development of an understanding of rate of change, derivative, and integral. The *rate of change* describes how fast or slow an object changes within some bounded interval, relative to the size of the interval. The *derivative* is the instantaneous rate of change of an object. The *integral* is the total change of an object. The students in the Calculus/Physics program encounter both the derivative and integral in four contexts: numeric, symbolic, graphical, and physical. The numeric, symbolic, graphical, and physical contexts have been discussed throughout the mathematics education literature (Zandieh, 2000; Aspinwall, et. al, 1997; Sullivan, 1995). I will review studies focusing on students’ conceptualizations of the various representations of calculus concepts in the next chapter.
Since I will refer to the numeric, symbolic, graphic, and physical contexts throughout the remainder of my discussion, I will define them at this time. The numeric context refers to a discrete presentation of data, such as in a table or chart. The symbolic context involves the actual algorithms used for computing, for instance, derivatives or integrals. Graphical contexts refer to the recognition and construction of graphs of mathematical objects and functions. Physical contexts include concrete examples of and experiences with mathematical objects and functions. These contexts become representations only when a student uses it to express a conception (Confrey, quoted in Cobb, Yackel, & Wood, 1992). These four representations can be thought of as components that constitute the concepts of derivative and integral. Other representations of calculus concepts have been introduced and discussed throughout the mathematics education literature, such as verbal representations and pictorial representations. However, for the purposes of this study, I will focus on only the four representations mentioned above: numeric, symbolic, graphical, and physical.

A student’s conceptualization of the different representations of derivative directly informs his or her concept image of derivative. An understanding of one representation can dominate a student’s concept image of derivative. When a student has a richer conceptualization of one representation over another representation, I will say that the student’s concept image is unbalanced. The goal of learning, then, can be thought of as a quest for a balanced concept image: a concept image informed by an equally rich conceptualization of all the representations of a certain concept.

To illustrate this, I will use a circle to represent the concept image. The circle will be divided up according to a student’s understanding of the various representations.
Figure 2 is an example of a student’s concept image that has a richer conceptualization of the algebraic and graphical representations of derivative than the physical and numeric representations. Since the student has a richer conceptualization of the algebraic and graphical representations of derivative, these pieces take up more of the circle, thus displaying an unbalanced circle. Figure 3 is an example of the “balanced” concept image of derivative; an understanding of one representation does not dominate any other representation. Thus the understanding of all the representations is balanced; this shows an equally rich conceptualization of each representation.

Since I am concerned with re-constructing students’ concept images, I am concerned with their mental activities. My hypothesis is that a student who has constructed detailed mental pictures of various representations of a concept and can readily draw upon these mental pictures has a richer conceptualization of that concept than a student who has failed to construct or constructed vague mental pictures of various representations of that concept and cannot easily summon those mental pictures. For example, a student who has a well-developed mental picture of the symbolic representation of derivative but a weak or vague mental picture of the graphic representation will probably succeed at tasks involving using the power rule to take derivatives of polynomial functions but may have trouble sketching the derivative and antiderivative graphs of \( f(t) = t^4 - 3t^2 + 7 \).

Recall that Tall and Vinner (1981) defined concept image as that which “describes the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes” (pp. 152). My focus in this study is on the mental pictures, although I recognize that the connections between
internal conceptualizations significantly contribute to an individual’s understanding of a concept. Many researchers have discussed the importance of students’ ability to translate between various representations of a mathematical concept. In particular, this idea of

![Diagram showing balanced and unbalanced concept images of derivative]

**Figure 2:** A student’s unbalanced concept image of derivative. This student has a stronger understanding of the algebraic and graphical representations than the numeric and physical representations.

![Diagram showing balanced representation of derivative and integral concepts]

**Figure 3:** The derivative and integral concepts are composed of numeric, algebraic, graphical, and physical representations.
translating between multiple representations has been explored extensively in the context of students’ understanding of function (Schwarz & Dreyfus, 1995; Janvier, 1987). However, for the scope of the present study, I chose to focus on students’ ability to construct multiple representations and not consider their ability to translate between those representations. Because of the complex and detailed nature of conducting research aimed at re-constructing students’ conceptualizations of mathematics concepts, I feel that the first step in understanding the role of physical representations in students’ conceptualizations of calculus concepts is to identify how the physical representation, along with the numeric, symbolic, and graphical representations make up the students’ concept image. A natural extension of this study is to examine the nature of the connections students make between internal representations and how those connections contribute to students’ concept images.

Additionally, the relationship between calculus concepts such as derivative and integral was used to guide the development of interview questions designed to probe students’ understanding of calculus concepts. A rich conceptualization of the concept of derivative, for example, does not only involve the representations of derivative, but also the relationship between derivative and other calculus concepts such as rate of change and integral. Furthermore, the concepts of derivative and integral incorporate other mathematical and physical notions, such as slope and area. For instance, a student’s ideas about slope will inform his or her understanding of rate of change, and an understanding of rate of change is an essential aspect of understanding the concept of derivative.
Finally, Nemirovsky and Rubin (1992) outline three broad assumptions about students’ understanding of the relationship between function and derivative which guide their work. I would like to restate these assumptions and extend them to include my thoughts about the relationship between function and anti-derivative and derivative and integral.

1. Most humans have some intuitive knowledge about the relationship between function and derivative. This knowledge may be an understanding of position and velocity, level and flow, or other rates of change. We have the capacity to generalize this context-specific knowledge to other situations involving change.

2. The relationships between function and derivative, function and anti-derivative, and derivative and integral are notions that always remain open to further elaboration. In particular, it is my belief that no person has a complete understanding of such relationships.

3. Students’ performance in solving problems involving the function/derivative, function/anti-derivative, and derivative/integral relationship is strongly affected by contextual parameters.

These assumptions helped focus the lens of my theoretical framework by more specifically situating the theories, definitions, and ideas that make up the framework. Furthermore these assumptions influenced the design of interview tasks in that many of the tasks focused on the relationship between function, derivative, and anti-derivative and many tasks were designed to elicit students’ context-specific knowledge.
Summary

This chapter discussed the three major components of my theoretical framework: concept image, representation theory, and constructivist learning theory. Maxwell (1996) describes a theoretical framework as:

A formulation of what you think is going on with the phenomena you are studying – a tentative theory of what is happening and why. The function of this theory is to inform the rest of your design – to help you to assess your purposes, develop and select realistic and relevant research questions and methods, and identify potential validity threats to your conclusions (pp. 25).

As I previously described, my framework explains how my research lens works, guides me in pointing my lens, and helps me focus my lens. The role of my theoretical framework in shaping the present study will be discussed further in Chapter IV. The next chapter, which discusses the relevant literature, helps situate my study within the context of existing mathematics education research.
CHAPTER III

LITERATURE REVIEW

Introduction

The theoretical framework for the present study was presented in the previous chapter. The theoretical framework, grounded in the theories of constructivism and representation, and guided by the notion of concept image, serves to shape the research questions and inform the methodology of the present study. The research questions must have a relationship to previously conducted research. A review and discussion of the relevant research will be presented in this chapter.

I will be drawing from both the physics education and mathematics education bodies of research in this literature review. The physics education literature and mathematics education literature can be viewed as two bodies of research which, in addition to serving the needs of their respective communities, complement the works of each other. I have found that many results in the mathematics education literature are replicated or underscored in the physics education literature. Whenever possible, I will highlight links between physics education and mathematics education research studies.

Research conducted in these areas provides a footing for my investigation since the work of previous researchers enables me to pinpoint pertinent issues and reflect on the work of previous investigations. Furthermore, the existing literature provides a context for which this investigation can be situated. Readers might notice that many studies are reports of qualitative investigations, such as case studies and teaching
experiments. Few studies serve to synthesize the literature, set the stage for future research, or build theory. I believe that future work should address the need for a synthesis of the research on calculus learning.

This literature review is organized around two overarching areas of research: mathematics education and physics education. I will begin with a discussion of the mathematics education literature, which I have arranged into three subsections: (1) Research on students' understanding of calculus concepts, (2) the role of prior experiences in understanding calculus concepts, and (3) the role of contexts in understanding calculus concepts. Then I turn to the physics education literature, which I also organized into three subsections: (1) the role of experience in physics education, (2) student difficulties with graphical contexts, and (3) the role of mathematics in learning physics.

Research on Student Understanding of Calculus Concepts

Overview

During the past two decades, investigations into student understanding of calculus concepts have become a focus of mathematics education researchers. For instance, the formation of the Association for Research on Undergraduate Mathematics Education (ARUME) in 1999 helped raise awareness of both mathematicians and mathematics educators to the body of existing research on undergraduate mathematics education.

Investigations into students' understanding of calculus concepts have shown that students are able to successfully carry out methods of differentiation and integration, but sometimes lack the conceptual underpinnings necessary to explain procedures, work
through problems using multiple strategies, and make connections between concepts (Orton, 1983a; Orton, 1983b; Vinner, 1989; Ferrini-Mundy & Graham, 1994; Norman & Prichard, 1994). Other research studies examine the effect of technology on students’ understanding of calculus concepts (Heid, 1988; Beckmann, 1990), the effect of reform efforts on students’ understanding of calculus concepts (Bookman & Friedman, 1994; Frid, 1994); Porzio 1997; Armstrong, Garner, & Wynn, 1994; Armstrong & Hendrix, 1999) and factors affecting student achievement in calculus (Edge & Friedberg 1984; Ferrini-Mundy & Gaudard 1991).

Since the late 1980’s the mathematics and mathematics education communities have undertaken major efforts to reform the way calculus is taught at the undergraduate level. For example, reports such as *Shaping the Future: New Expectations for Undergraduate Education in Science, Mathematics, Engineering, and Technology* (National Science Foundation, 1996) and *Calculus for a New Century: A Pump not a Filter* (Steen, 1988) have called for a reform of undergraduate science and mathematics courses. National Science Foundation-funded curriculum programs such as the Harvard Calculus Consortium have produced textbooks and classroom materials to “encourage students to think about the geometrical and numerical meaning of what they are doing” (Hughes-Hallet, Gleason et al., 1994, pp. vii).

Armstrong and Hedrix (1999) reported on the results of a study comparing student achievement in three calculus programs: traditional calculus, Harvard Consortium Calculus (CCH), and Calculus using Mathematica (CUM). The authors found that there was no statistically significant difference in student performance in post-calculus courses between students who completed a two-semester traditional, CCH, or CUM calculus
sequence. The authors claim that since the students in the reform calculus classes were learning the skills necessary for success in subsequent classes, students were better served by the reform classes because of the unseen advantages of reform-type classes. These advantages include exposure to computer software, group learning, and the opportunity to work on complex projects.

Studies such as Armstrong and Hendrix’s (1999) heed the call for more research on the effects of technology, group learning, and reform curricular materials on students learning (Ferrini-Mundy & Graham, 1991). Ferrini-Mundy and Graham (1991) have also called for researchers to investigate the effects of introducing substantial physical applications in the calculus course and whether or not physical examples help in the learning of calculus concepts (pp. 633). I believe the results of the present research study begin to answer the questions posed by Ferrini-Mundy and Graham.

Although my research study focuses on college-level students’ conceptualizations of calculus concepts, these students have been formally and informally developing notions of rate of change for some time. Many researchers have suggested that rate of change ideas can be developed as early as elementary school (Thompson, 1994; Turner, Wilhelm, & Confrey, 2000). Additionally, the notions of ratio and proportion are the underpinnings of the concept of rate of change (Lesh, Post, & Behr, 1988; Arons, 1990). For this reason, I briefly discuss the major results of research on students’ understanding of ratio and proportional reasoning.

**Ratio and Proportional Reasoning**

The idea of rate of change can be thought of as an extension of the concepts of ratio and proportion. Thus, research on children’s understanding of ratio and proportion
lays a foundation for research on older students' understanding of rate. Arons (1990) claims that, "One of the most severe and widely prevalent gaps in cognitive development of students at secondary and early college levels is the failure to have mastered reasoning involving ratios" (pp. 3).

Tourniaire & Pulos (1985) extensively reviewed and discussed the proportional reasoning literature up to 1985. They found that most studies discuss one of two basic types of successful strategies in solving proportions: multiplicative and building-up. Multiplicative strategies involve relating terms within one ratio multiplicatively and then extending the relation to a second ratio. Building-up strategies involve establishing a relationship within one ratio and extending it to a second ratio by addition. Building-up strategies work easily with simple problems, but become difficult and cumbersome to use when problems contain non-integer ratios.

Tourniaire & Pulos also suggest that errors in solving proportional reasoning problems arise from using an inappropriate strategy to solve the problem or misusing a correct strategy. The authors indicate that variability in student performance on proportional reasoning problems can be attributed to a number of factors related to the problem context. These factors include the type of problem, the presence of discrete or continuous quantities, and familiarity of the problem context.

More recent research supports the findings in the studies cited by Tourniaire and Pulos (1985), and has shown that many students still tend to favor a building-up approach to proportional reasoning problems (Kaput & West, 1994; Simon & Blume, 1994). Recent studies have also investigated the effect of task and context factors on students' ability to solve ratio and proportion problems.
For example, Lawton (1993) investigated the contribution of contextual factors to student errors in solving proportional reasoning problems. She found that the college students in her study solved more proportional reasoning problems correctly when the contents of the items in the problem were distinct from one another. Lawton claims that, "A proportional relationship involves the 'translation' of units of one item into units of another; this translation process is more readily triggered if the items are seen as being substantially different from each other" (pp. 465). Thus, students are more apt to make conceptual errors if the items in the problem exhibit physical similarities.

Finally, a substantial body of research has been developed addressing adults' ability to solve mathematical problems in everyday life situations (Nunes, Schliemann, & Carraher, 1993; Hoyles, Noss, & Pozzi, 2001). These studies suggest that adults proficiently solve proportional reasoning problems using informal strategies, many of which are situation-dependent. Furthermore, these informal strategies used by adults in everyday situations do not resemble any school-taught methods of solving proportional reasoning problems.

I have discussed a small portion of the extensive literature in the domain of ratio and proportion. This review of the ratio and proportion literature serves to highlight the complexity and importance of these concepts. The next section briefly discusses another fundamental concept in calculus learning, the concept of function.

The Role of Function in Introductory Calculus

The concept of function has been studied by many researchers in a variety of contexts due to the complex nature of this mathematical construct (see Thompson, 1994 for a synthesis of the functions literature and its relevance to undergraduate mathematics...
curriculum). In particular, researchers found that students' underdeveloped conceptualizations of the function concept accounts for some of the difficulties that students encounter in the calculus class (Breidenbach, Dubinsky et al., 1992; Thompson, 1994a).

Research shows that students tend to think prototypically about functions (Dreyfus & Vinner, 1989; Tall, 1992). For example, if a student experiences functions only in a symbolic form, then the student may believe that a relation is a function only if it can be assigned a formula (Tall, 1992, pp. 498). Similarly, some students assume that continuity is a necessary component of a function or that a complicated graph cannot be a function since many students encounter only continuous graphs of functions (Vinner, 1983; Markovits, Eylon, & Bruckheimer, 1988; Dreyfus & Vinner, 1989).

Carlson's (1998) investigation of undergraduate and graduate students' conception of function indicates that the function concept develops slowly over time for many students. Furthermore, Carlson reported that many students could not retrieve basic information about functions to solve non-routine mathematics problems. Similarly, Orton (1983a) found that students working on derivative task problems could not recall rate of change ideas. I will discuss Orton's (1983a) study in more detail below.

Finally, Thompson (1994a) highlights two perspectives on defining functions: the correspondence and covariational approaches. Thompson points out that, although both the correspondence and covariational approaches are meaningful ways to understand functions, only the correspondence approach has been considered in K-14 curriculum. The covariational approach, Thompson claims, is crucial for understanding the concept of rate.
Rates of Change

The notion of rate of change is closely linked to the concept of function since a rate of change is often algebraically described by a function. In particular, the importance of covariation as a link between students’ understanding of function and rate has been discussed by Confrey (Confrey, 1994; Confrey & Smith, 1994) and Thompson (Thompson 1994a, 1994b; Saladanha & Thompson, 1998). The works of both Confrey and Thompson have not only underscored the importance of developing rate concepts at an early age, but also have uncovered students’ early intuitions and misconceptions about rate. These studies provide one way of examining how students develop the grounding for an understanding of rate of change.

Confrey and Smith (1994) distinguish two approaches to developing the concept of function: the correspondence approach and the covariation approach. The correspondence approach, which Confrey and Smith claim is most prevalent in the current curriculum, is a rule-based approach to functions. The notion of function is developed as a rule to determine output values from unique input values. Thus, one produces a correspondence between the input and output values, which is conventionally denoted $y = f(x)$ (pp. 137). Alternatively, the covariation approach coordinates movement between input and output values, focusing on the change between output and input values, rather than finding a formula to describe how to obtain the output value from a given input value. Confrey and Smith (1994) describe a student using the covariation approach as follows:

Students working in a problem situation first fill down a table column with x-values, typically by adding 1, then fill down a y-column through an operation they construct within the problem.
context. Such an approach has the benefit of emphasizing rate-of-change (pp. 135).

The authors claim that the covariation approach makes the concept of rate of change more visible to students. Furthermore, Confrey and Smith (Confrey & Smith, 1994; Smith & Confrey, 1994) assert that the covariational approach to function was central in the development of both the exponential and logarithmic functions — functions that often play an important role in the calculus curriculum.

While Confrey’s work focuses on developing an understanding of students’ rate constructs and clarifying the differences between rate and ratio, she also makes claims about students’ intuitions that are relevant to my study. In particular, Confrey claims that in her work with students’ understanding of rate, she has witnessed evidence of a ‘primitive’ understanding of rates of change based on students’ experiences. In particular, Confrey and Smith state, “Volume turned up, running hard to end a race, breathing slowing down after rest, are all rate concepts to children” (pp. 156). Thus, students’ experiences with the physical world provide them with a precursory understanding of rate of change concepts.

Thompson (1994a) describes a three-part teaching experiment which focused on (1) probing and extending the student’s conception of speed, (2) extending the student’s conception of speed to include the conventional notion of average speed, and (3) extending the student’s conception of speed to a more general concept of rate (pp. 198). Thompson used a computer microworld, Over and Back as a means for assigning tasks to the student throughout the teaching experiment. The student, JJ, held an initial conception of speed as distance but throughout the teaching experiment moves to a
conception of speed as a ratio. The idea that students think about speed as distance has been documented previously by Thompson (Thompson & Thompson 1994, 1992).

Thompson has written extensively about the teaching, learning, and understanding of the concept of rate at various levels (Thompson 1994a, 1994b; Thompson & Thompson 1994, 1992). Thompson’s work with advanced undergraduate and graduate students (1994b) in a class focused on using computers in teaching mathematics, showed that students’ “fixation on accrual as a solitary object” parallels JJ’s difficulties conceptualizing speed as a rate of change. Thompson noticed that both the older students, as well as JJ, had difficulties thinking about speed as a covariation.

Thompson did not propose that the older students had the same understanding of speed as JJ, but rather that they had a ‘weak scheme’ for the concept of average rate of change. Thompson defines average rate of change by noting that, “...if a quantity were to grow in measure at a constant rate of change with respect to a uniformly changing quantity, then we would end up with the same amount of change in the dependent quantity as actually occurred” (pp. 269). Thompson found evidence that the older students did not have a conceptualization of average rate of change as defined above and thus they had difficulty justifying a covariational approach to rate.

Thompson’s research on advanced undergraduate and graduate students’ understanding of rate is one of the few studies that focuses on upper-level students’ understanding of rate outside of a calculus class. Typically, issues involving students’ understanding of rates of change are part of larger studies examining students’ understanding of the major calculus concepts (derivative, integral, differential equations). Studies such as Thompson’s (1994a, 1994b) and the work of Confrey (1994) have shed
some light on questions concerning students' initial conceptions of rate and average rate of change, and the issues and challenges involved in teaching these topics to students of various ages.

Equally important are those studies that examine students’ conceptualizations of rate of change before, during, and after calculus instruction. Specifically, these studies help us understand the effects of schooling on students’ conceptualizations of rate of change. The results of these studies can and should inform curricular development and instruction.

Hauger (1997) discussed how precalculus students resolved errors in solving problems dealing with rate of change. Hauger reported on in-depth interviews he conducted with four precalculus students. The four students all initially exhibited the belief that a straight line on a graph represented variable rate. Each student, either on his or her own or with prompting from the author discovered that a curved graph represents variable rate. Hauger noted that the students relied heavily upon the shape of the graph to inform their understanding of varying rate of change. Furthermore, the students also considered comparing changes in intervals to determine if an object was traveling at a constant or a varying rate of change. Hauger’s conclusions help to give some sense of how students think about rates of change before calculus instruction.

Bezuidenhout (1998) investigated first-year university students’ understanding of the concept of rate of change. Bezuidenhout found that many students exhibited an inadequate intuition about the concept of rate of change that resulted in a confusion of the notions of ‘average rate of change’ and ‘arithmetic mean’. Many students interpreted the concept of ‘average rate of change’ as synonymous with ‘arithmetic mean’.
Bezuidenhout concluded that in many cases, students' ideas about arithmetic mean dominated their understanding of average rate of change.

Bezuidenhout also noticed that students tended to include the derivative somewhere in their definitions of average rate of change. Students in her study did not seem to understand the difference between instantaneous rate of change and average rate of change. Furthermore, students were unwilling to modify their ideas about and methods of calculating average rate of change even when they encountered situations in which their own methods did not work.

Similarly, Thompson (1994b) noted that students confused the notions of difference quotient and derivative. Thompson investigated advanced undergraduate and graduate students' understanding of rate within the context of the Fundamental Theorem of Calculus. What is striking about Thompson's study is the level of the student involved; these students were mostly mathematics majors, having taken three semesters of calculus and many had taken or were enrolled in an advanced calculus class at the time of the study.

Thompson highlighted an episode in which a pair of students were having difficulty interpreting the function: \( r(x) = \frac{d(x + 0.1) - d(x)}{0.1} \), where \( d(x) = 16x^2 \) and represents the distance an object falls \( t \) seconds after being released. The students initially believed that \( r(x) \) represented how fast an object fell during some tenth of a second. Only after probing by Thompson did the students realize that \( r(x) \) represented the average speed of the object during some tenth of a second. Thompson stated:

Those students who experienced difficulty seemed to want to think of the difference quotient as "the derivative" and interpret it as 'how fast it [the function] is changing," without interpreting the
details of the expression as an amount of change in one quantity in relation to a change in another (pp. 246).

Orton (1984) also maintained that students often confuse the notions of average and instantaneous rate of change. "It seems...we must be careful that we do not assume too much in terms of pupils' abilities to sort out important ideas like variable speed, constant speed, average speed, and speed at an instant" (p. 24). Orton suggested that students have difficulties with the concept of rate of change because rate of change involves concepts related to proportionality and ratio and proportion concepts present difficulties for many students.

Using an interview format to collect data from high school and college-level students, Orton (1984) found that over one-third of the students interviewed could not correctly answer a question involving finding the average rate of change between two points on a graph. Furthermore, one-half of the students interviewed incorrectly answered or could not attempt to answer a question involving finding the average rate of change between two points with the same y-values. Orton's (1984) findings suggest that many students possess an underdeveloped understanding of rate of change.

Hauger (1995) discusses the tools and resources that students bring with them to solve rate of change problems and correct errors in their work. Hauger reported on the results of a study that focused on the strategies that students use to solve both average rate of change and instantaneous rate of change problems. Hauger grouped the students' strategies for solving rate of change problems into three clusters: global rate of change strategies, interval rate of change strategies, and point-wise rate of change strategies. Interval rate of change corresponds to the more universal term 'average rate of change'; likewise point-wise rate of change corresponds to 'instantaneous rate of change'. Global
rate of change knowledge deals with general properties of a function such as increasing and decreasing.

Hauger found that the calculus and post-calculus students generally gave more detailed descriptions of the global features of a graph than the precalculus students. This result is not surprising since the study of calculus involves analyzing extrema and points of inflection of graphs and determining where functions are increasing, decreasing, concave up, and concave down. Hauger's results concerning students' methods of solving average rate of change problems indicated that most students referred to slope to solve the problem or they examined the vertical change of the y-variable.

These research studies show that although students do possess intuitive conceptions of rate of change, these intuitions are often ignored both leading up to and during calculus instruction. Confrey (Confrey 1994; Confrey & Smith, 1994) and Thompson (1994a) have proposed a covariational approach to functions in order to more closely align the mathematical formulation of rate of change with students' intuitions about the concept. The results of work done by Bezuidenhout (1999), Thompson (1994b), Orton (1983a), and Hauger (1995, 1997) show that students all too often confuse their own intuitive ideas about rate and speed and notions of fundamental concepts such as average and slope that they learned in the classroom, with the formalization of rate of change in calculus. Students' experiences, both formal (in the classroom) and informal (everyday experiences) will shape how and what they learn in calculus.

The Derivative Concept

Students' understanding of the concept of derivative has been investigated in a number of studies in the past few decades. A study conducted by Orton (1983a) showed
that students generally were able to carry out computational differentiation tasks, but had considerably more difficulty with problems addressing a conceptual understanding of derivative or a graphical approach to the derivative. More recent studies (e.g., Ferrini-Mundy & Graham, 1994; Aspinwall, Shaw, & Presmeg, 1997) have supported Orton’s results.

Orton (1983a) used an interview format to collect data from 110 high school and college students. Orton administered a number of tasks designed to probe students’ understanding of the derivative and its applications. The results of Orton’s study indicate that students are capable of carrying out computational differentiation of functions, but had considerable difficulty solving problems related to average and instantaneous rate of change and differentiation as a limiting process. For example, Orton found that 74 of the 110 students did not answer or incorrectly answered questions that asked students to find the average rate of change of a function between the points $a$ and $a + h$ and then relate their answer to finding the instantaneous rate of change at a point.

Ferrini-Mundy and Graham (1994) conducted a series of interviews with first-year calculus students over a two-semester calculus course. Ferrini-Mundy and Graham found that for one student, who exhibited proficiency in computing derivatives using algorithms, the same student could not explain the relationship between a function and its derivative or how the tangent line relates to the derivative. Furthermore, the student demonstrated a poor understanding of the geometric representation of derivative.

Aspinwall, Shaw, and Presmeg (1997) reported on the results of a case study they conducted with a student, Tim. Tim was able to compute the derivatives of functions using rules and procedures he learned in calculus class. However, Tim’s understanding
of the graphical context of derivatives was shaky. Furthermore, Tim was unable to translate between graphical and symbolic contexts of functions without prompting from the interviewer.

While Orton (1983a), Ferrini-Mundy & Graham (1994), and Aspinwall, Shaw, & Presmeg’s (1997) studies answered questions about how and why students come to understand derivative, other studies focused on developing a framework for organizing and classifying student conceptualizations of derivative.

Zandieh (2000) developed such a framework using the ideas of multiple representations and three process-object pairs: ratio, limit, and function. Zandieh developed her framework by gathering information about how the mathematical community talks about the concept of derivative at the first-year calculus level (pp. 104). The framework is useful for describing students’ understanding of derivative.

Zandieh uses a matrix or grid to organize the results of her finding for a particular student. The matrix lists contexts (graphical, verbal, physical, symbolic, and others) as column headings and process-object layers (ration, limit, function) as row headings. Then, based on students’ answers to interview questions, the entries of the matrix may or may not be filled in, denoting a type of understanding of that row and column intersection.

Snook (1997) also developed a framework for organizing and synthesizing students’ understanding of derivative. Snook’s focus, in contrast to Zandieh’s, was to compare students’ written performance on derivative tasks with their verbal performance on derivative problems during talk-aloud interviews. Snook developed a Combined Model of Understanding framework to analyze her data.
The Integral Concept

Research on the integral concept has received considerably less attention than research on derivatives and rates of change. I report here on the studies that have significantly contributed to the domain of research on integration.

Orton (1983b) investigated both high school and college-level students’ understanding of integration. The results of this study revealed that the students who participated in this investigation exhibited difficulty understanding integration as the limit of an infinite sum. For example, Orton found that although most students were able to correctly calculate the area under the curve $y = 2x - x^2$ from 0 to 3 in two pieces, many of these students could not explain why the integral must be calculated in two separate pieces. Some students gave the response, “That’s the way we were taught to do it in school” (pp. 8).

Orton expressed optimism that increased development and use of the calculator will aid in students’ understanding of fundamental calculus concepts. However, Ferrini-Mundy and Graham’s (1994) case study of a calculus student, published eleven years after Orton’s (1983b) study showed that even with an increase in technological developments, such as the widespread use of graphing calculators, students still exhibit difficulties understanding fundamental calculus concepts.

Ferrini-Mundy and Graham (1994) reported that students in their study interpreted the integral as an indication to perform a task. One student that the authors interviewed exhibited proficiency in computing integrals using algorithms, but the same student’s ability to interpret the integral was weak. Furthermore, the student was not able to explain any relationship between limits and integration.
The Role of Prior Experiences in Understanding Calculus Concepts

Throughout the literature, researchers have alluded to the importance of prior experiences on students' conceptualization of calculus concepts (Thompson 1994a, 1994b; Hauger 1997; Nemirovsky & Noble, 1997; Noble, Nemirovsky, Wright, & Tierney, 1998; Speiser & Walter 1996). These experiences refer to both mathematical and non-mathematical episodes, and situations encountered both in and out of the classroom.

Nemirovsky & Noble (1997) used a computer-based tool, the Contour Analyzer, to probe one student’s understanding of slope. The Contour Analyzer creates height vs. distance and slope vs. distance graphs. Nemirovsky and Noble describe various episodes from their case study with the student, Karen, that illustrate how Karen’s prior experiences both in and out of a mathematics classroom shape what and how she learns.

For instance, Karen grapples with justifying why a graph of height vs. distance that has negative slope would correspond to a function drawn below the x-axis on a slope vs. distance graph. Karen struggles with trying to use a definition of slope as rise over run as a justification, but then makes a connection to her previous experiences using a motion detector in physics class in which she measured negative velocity. “Karen’s recognition that the sign of the slope graph showed the up/down slantiness of the board [of the Contour Analyzer] was grounded in her previous experience with velocity graphs and the motion detector in her physics class at school” (pp. 117).

Furthermore, Nemirovsky and Noble trace Karen’s struggle with the idea that a velocity graph could become more negative even if the object was accelerating to her experiences with counting negative numbers. Karen seemed to view the counting of
negative numbers as an illusion of something getting bigger. Karen recalled her past experience with negative numbers as an unusual situation. Thus, Nemirovsky and Noble suggest that the moments of remembering were not retrievals of information, but a reawakening of past experiences.

These moments of remembering were not mere retrievals of information, but were moments where the feeling or sense of past experiences was reawakened. These moments occurred at times of puzzlement, such as...when Karen suddenly came upon a memory of another experience with positive and negative graphs that had been puzzling to her, that of using a motion detector to create negative velocity graphs, which also reminded her of counting negative numbers. The common thread connecting velocity graphs, counting negative numbers, and the negative slope graphs Karen encountered in this interview was not a general mathematical property of signed quantities, but the experience of being perplexed by the illusion that something can be ‘getting bigger’ when it is also decreasing (pp. 125).

Nemirovsky and Noble illustrate that past experiences can powerful means to connect new experiences with prior knowledge. Furthermore, the connections are not made by a mere retrieval of information, but rather through a re-living of past feelings or sensations.

A series of articles by Speiser & Walter (1994, 1996) describe episodes from their own classrooms in which they used a sequence of time-lapse photographs of a cat’s motion to motivate and explain the concept of derivative (Speiser & Walter, 1994, pp. 135). Their story tells how using this teaching tool helped shape their thinking about the teaching and learning of differential calculus. Speiser and Walter (1996), reflecting on their experiences using the cat photographs to explore the derivative concept, claim that “…personal experience is part of how we understand and use our mathematics” (pp. 370). Speiser and Walter highlight the need to listen to students’ voices and ideas in the classroom.
Additionally, students' past experiences are an important component of a constructivist learning theory. As described in the previous chapter, when students encounter new concepts, they will attempt to assimilate those concepts into their already existing realm of experiences. Although the negative impact of students' prior experiences has been documented in the physics education literature (Clement, 1982; Halloun & Hestenes, 1985b), this possibility has been explored very little in mathematics education research. As I will discuss in an ensuing section, the physics education literature has shown that students' prior experiences may mislead them to possess naïve beliefs about motion (McClosky, Caramazza, & Green, 1980; Galili & Bar, 1992).

The Role of Contexts in Understanding Calculus Concepts

The importance of students' ability to work with calculus concepts in multiple contexts or representations has been addressed by a number of researchers (Stump 1998; Zandieh 1998; 2000). In a previous section, I discussed Zandieh's (2000) framework for classifying student understanding within multiple contexts. In this section, I will discuss research that further elaborates on the effect of context on students' conceptualizations of calculus concepts and end with a note of caution expressed by Thompson (1994a).

White and Mitchelmore focused on students' conceptualization of variable in their 1997 study of students' conceptual understanding of derivative. The researchers presented first year university calculus students with four different versions of four problems before, during, and after the calculus course. Each of the four versions of a problem was structured so that the manipulation required to solve each version was
basically the same (pp. 83). The difference between the four versions of a problem was
the amount of translation to symbols required.

White and Mitchelmore found that students generally had difficulty solving
problems where they were required to translate to an appropriate symbolization. The
authors reported that students' lack of conceptual understanding of variables caused them
to discount problem-specific contextual meaning as a way to treat variables. The authors
propose, "Students showing the manipulation focus have a concept of variable that is
limited to algebraic symbols; they have learned to operate with symbols without any
regard to their possible contextual meaning" (pp. 91). The authors theorize that a
conceptual understanding of variables and algebraic manipulations of variables is a
necessary precursor to the conceptual understanding of calculus concepts. Students who
hold an "abstract-apart" concept of variable, that is, a concept of variable limited to
algebraic symbols, only understand the process of applying rules to manipulate variables.

Oliveros and Santos-Trigo (1997) reported on the results of their investigation of
students' activities in a Grade 12 problem-based calculus class. The authors were
interested in documenting student roles in small and whole group discussions and
identifying when students exhibited difficulties in their conceptualization of rate. An
interesting result that the authors reported dealt with the students' interpretation of
problem situations when the data was given to them in different contexts.

For example, the authors presented pairs of students with data that described the
growth of a tumor and asked the students to pose and respond to three questions. The
data was given to the students in three different contexts: a table, a graph, and
symbolically. The researchers found that some students believed the information to come
from three different situations, a unique situation corresponding to each context of data presentation.

Porzio (1997) conducted a study examining the effects of three different instructional approaches to calculus on students’ understanding of numerical, graphical, and symbolic contexts. The three types of instruction examined were a traditional approach, one integrating the use of graphics calculators, and Calculus & Mathematica approach. Porzio gathered data through a post-test and one-on-one interviews with students.

Porzio found that the students enrolled in the class using a graphics calculator approach to calculus instruction tended to proficiently use graphical representations to solve problems but exhibited difficulty using symbolic representations or making connections between graphical and symbolic representations (pp. 5). More troubling, Porzio reported that a common difficulty among the graphic calculator students was a lack of understanding of the connection between the first derivative of a function at a point and its slope at that point (pp. 7).

Porzio attributes the graphic calculator students’ lack of understanding of the connections between the derivative representations to a lack of adequate time for reflective abstraction. That is, the students were not given enough opportunity in class to make connections between the representations. Furthermore, the students reported that they believed that the focus of the class was on learning calculus in a graphical context only; this belief, coupled with the lack of time to reflect on making connections, could lead students to compartmentalize their knowledge.
Sullivan (1995) conducted a study that supports claims about students’ compartmentalizing their knowledge into various representations of derivative. Sullivan evaluated a curriculum that focused on the numeric, graphic, and symbolic contexts of derivative. She found that the students in her study generally viewed each context separately and rarely used information from one context to aid in solving a problem in another context. Additionally, Sullivan reported that the students seemed to prefer working with the symbolic representation of derivative.

Some researchers warn against the use of multiple representations as a framework for which to situate students’ conceptualizations of mathematical concepts. In particular, Thompson (1994a) cautions that a missing element in research on representations is the idea of ‘representation’, itself. “Tables, graphs, and expressions might be multiple representations of functions to us [mathematicians and mathematics education researchers], but I have seen no evidence that they are multiple representations of anything to students” (pp. 39). Instead, Thompson claims that contexts such as graphs, formulas, and tables should be thought of as *representable*. As I outlined and discussed in the Theoretical Framework (Chapter II), I believe that graphs, formulas, and tables are contexts that become representations only when students use them to re-present their internal conceptions. Thus, I feel that my position on representations, although not completely aligned with Thompson's position, addresses his concerns.

**Research on Student Understanding of Physics Concepts**

The physics education literature provides an extensive collection of work on student understanding. I have concentrated this part of my literature review on those
studies from physics education that deal with students’ understanding of kinematics and
dynamics concepts. I chose to focus my review of the physics education literature on
kinematics and dynamics since in this study I will be focusing on the connections
students make between calculus and physics as they study kinematics and dynamics.

The physics education literature is composed of studies involving both
quantitative and qualitative research methods. Since physics courses are typically taught
at the high school and college levels in the United States, most of the physics education
research focuses on high school and college age students. Exceptions include, Galili and
Bar (1992) who studied students as young as 10th grade. Lawrenz (1986) and Kruger et.
al. (1992) who both conducted separate studies of elementary teachers’ understandings of
physics concepts.

Many of the physics education studies classify and discuss students’ difficulties
with and misconceptions of physics concepts. These studies also identify a basis for
students’ misconceptions of physics concepts. The results of such studies helped
pinpoint and classify the physics misconceptions of students in this study. Once I
classified a students’ misconception, I was able to trace how that misconception
influenced his/her conceptualization of a related calculus concept, based on the physics
education literature.

students’ understanding of the concepts of velocity and acceleration. The authors found
that students tended to confuse the concepts of position and velocity on interview speed
comparison tasks. The authors attribute this position and velocity confusion to the
students' inability to connect the concept of velocity with their interpretations of real-world phenomena (Trowbridge & McDermott, 1980, pp.1027).

Similarly, Trowbridge and McDermott (1981) found that students often confuse the concepts of velocity and acceleration. The authors report that students lacked an understanding of acceleration as the ratio of change in velocity over change in time. The results of Trowbridge and McDermott's (1980, 1981) studies have been replicated in other research studies (Peters, 1982; McDermott, Rosenquist, & van Zee, 1987).

Research has also shown that students exhibit difficulties making legitimate connections between force and motion (Clement, 1982; McDermott, 1984; Galili & Bar, 1992). For example, many students enter introductory physics classes holding the belief that force is necessary to sustain motion at any speed (McDermott, 1984) and that in the absence of force, an object is either at rest or slowing down (Clement, 1982). McDermott (1984) suggests that students possess such strong beliefs about force and motion since these beliefs are validated in everyday life experiences (pp. 28). The idea that students' life experiences shape how and what they learn is prevalent in the physics education literature and is the next topic I will discuss in detail.

The Role of Experience in Physics Education

Studies examining students' understanding of introductory physics concepts have resulted in revealing the dominant role of experience in students' learning (Clement 1982; Halloun and Hestnes 1985; Arons 1990; Thornton 1992; Thornton 1997; McDermott, Shaffer et al. 1994). For example, Trowbridge and McDermott (1980) used revised Piagetian motion tasks in order to probe students' understanding of one-
dimensional velocity. The authors found that students who failed at speed comparison tasks did so because they used a position criterion to determine relative velocities. The authors hypothesized that students who used position criterion to determine relative velocity did so because they could not bridge the gap between their observations of the world and the concepts underlying kinematics.

Our study has shown that prior to instruction the student typically has a repertoire of procedures, vocabulary, associations, and analogies for interpreting motion in the real world. These, taken together, may be considered as a set of protoconcepts which antedate understanding of the concepts of kinematics. Often students fail to make connections between these two sets of ideas. For example, as our investigation demonstrates, students frequently do not relate their intuition of how fast an object is going to the ratio of the distance traveled to the elapsed time or to the idea of velocity at an instant (p. 1027, italics in original).

Additionally, Trowbridge and McDermott showed that for some students, these persistent misconceptions of velocity seemed to remain even after several weeks of instruction. Trowbridge and McDermott's work not only shows that students bring their experiences from the world with them to the classroom, but that students' experiences may hinder understanding of certain physics concepts.

Goldberg and Anderson (1989) documented students' difficulties working with graphical representations of negative values of velocity. The authors concluded that many students have difficulty working with negative velocity because they are thinking only of speed, thus ignoring the directional component of velocity. One reason Goldberg and Anderson give as a possible explanation for students disregard for the directional element of velocity is that students encounter speed in everyday life situations and do not necessarily focus on the direction of the speed.
In everyday life students are familiar with the magnitude of velocity, namely speed. Although they may recognize, through their coursework in physics, the directional aspects of velocity, everyday usage may cause them to think of the magnitude and direction as completely separate attributes that need not be combined in one graphical representation (pp. 258).

Research on student understanding of mechanics concepts also reveals that students often misinterpret fundamental concepts of motion because they are influenced by a strong Aristotelian\(^1\) view of the world. That is, students’ experiences in the world inform what they perceive to be true, and what students believe to be true often goes against the principles of Newtonian physics.

Champagne, Klopfer, and Anderson (1980) investigated the factors influencing students’ difficulties in learning mechanics concepts. The authors found that many students enter an introductory physics course with some intuitive notions about how objects move.

Each student usually has a rich accumulation of interrelated ideas that constitute a personal system of common-sense beliefs about motion. These common-sense intuitive ideas, based on years of experience with moving objects, serve the students satisfactorily in describing the world. Nevertheless, this belief system is quite different from the formal system of Newtonian mechanics that the physics course seeks to teach. To a large degree, the rules of the belief system [of the students] parallel the descriptive aspects of Aristotelian physics. The Newtonian paradigm appears esoteric and unfamiliar to the uninitiated students in comparison with the comfortable and intuitive Aristotelian paradigm (pp. 1077).

Additionally, the authors found that even those students who had a year of high school physics still held onto Aristotelian beliefs about motion.

\(^1\) Champagne, Klopfer, & Anderson (1980) refer to students’ common sense beliefs about motion as Aristotelian, or following the principles of Aristotelian physics. In particular, Aristotelian physics stands in contrast with Newtonian physics.
Halloun and Hestenes (1985) and Clement (1982) found that even after formal instruction on Newtonian mechanics concepts, students still possess an Aristotelian outlook on motion. Halloun and Hestenes (1985) claim that “students are not so easily disabused of common sense beliefs, because their own beliefs are grounded in long personal experience.” These results are not just limited to the population of beginning college students. Galili and Bar (1992) investigated this phenomenon with a range of student from 10th graders to preservice teachers. Galili and Bar again found that students tend to hold onto preconceived notions of motion, even after formal instruction.

Students’ Difficulties with Graphical Contexts

Research on student understanding of kinematics reveals that students often exhibit difficulties when interpreting graphs of motion, velocity, and acceleration (Trowbridge and McDermott 1980; McDermott, Rosenquist et al. 1987; Goldberg and Anderson 1989; Beichner 1994). As discussed above, Trowbridge and McDermott (1980) found that students who failed at speed comparison tasks did so because they used a position criterion to determine relative velocities. This phenomenon was recorded by a number of other physics education researchers (Peters, 1982; Goldberg & Anderson, 1984; Reif & Allen, 1992) and has also been recorded in the mathematics education literature (Clement 1989; Leinhardt 1990; Hauger 1997).

In particular, Peters (1982) showed that honors students in an introductory physics course exhibit the same types of misconceptions as non-honors students do. Peters found that some of the honors students in his study showed no conceptual distinction between the concepts of position and velocity (pp. 502). Other students in his study exhibited
errors that are also well-documented in the literature such as inability to work with negative velocity and a tendency to draw the velocity vs. time graph resembling the shape of the position vs. time graph.

McDermott extended the 1980 study she conducted with Trowbridge in order to further explore student difficulties with kinematics concepts. McDermott, Rosenquist, and van Zee (1987) examined students' errors interpreting and producing graphs. The authors uncovered many sources for student difficulties linking the graphical representation with the physics concept. First, the authors noted student difficulties discriminating between the slope and the height of a graph. Students confused the information given by the height of a position versus time graph with the slope of the curve of a position versus time graph.

The second difficulty McDermott and her colleagues noticed was students' inability to distinguish between changes in height and changes in slope. When faced with problems in which they must identify where the motion of an object is slowest or fastest on a graph of position versus time, students often interpret the height of the graph as an indicator of motion of the object. The authors claim that, "Instead of looking for changes in slope, many students focus on the more perceptually obvious changes in height" (pp. 505). Again, this phenomenon of confusing height and slope has been documented in the mathematics education literature (Orton 1983, 1984).

The third difficulty McDermott and her colleagues documented was the relationship between a graph of a function and its derivative. McDermott and her colleagues found that often students were unable to produce a graph of velocity versus time when given a graph of position versus time. The authors noticed that many times...
students' graphs of the derivative of a function closely resembled the graph of the function. Additionally, McDermott and her colleagues reported that many students had difficulty interpreting area under a curve. In particular, the authors reported that the students found tasks involving the interpretation of area under a velocity versus time graph difficult because they cannot visualize the motion that is depicted in the velocity versus time graph (pp. 506). Similarly, Nemirovsky & Rubin (1992) found that when they administered problems in which students were required to sketch velocity graphs given position graphs, the students drew graphs of velocity that resembled the position graph in sign (positive/negative) and movement (increasing/decreasing) rather than using the relationship between position and velocity to sketch a graph of velocity.

Finally, McDermott and her colleagues investigated errors students make when connecting graphs to the real world. Some of these errors include representing continuous motion by a continuous line, the inability of students to separate the shape of a graph from the path of motion, and the inability of students to distinguish among different types of motion graphs. Some of these observations have also been noted in the mathematics education research. In particular, Dugdale (1993) reported on her observations that students inappropriately convert information from problems into features of graphs.

The physics education literature provides a distinct perspective on student understanding that critically informs this study. Not only does this literature provide a window into an additional facet of students' understanding of calculus-based concepts, but it also complements much of the research conducted on students understanding of calculus concepts by mathematics educators.
The Role of Mathematics in Understanding in Physics

Researchers in physics education have investigated the relationship between students’ mathematical ability and students’ performance in physics courses. Cohen, Hillman, and Agne (1978) found that SAT mathematics scores correlate highly with the level of physics course and the final grade in the course. Other researchers have shown that mathematical skill is one of many factors necessary for success in physics (Hudson & McIntire, 1977; Champagne, Klopfer, & Anderson, 1980).

Wittmann investigated students’ understanding of waves in his 1998 dissertation. In particular, Wittmann investigated the physical interpretation of the mathematics that describe propagating waves (pp. 55). Wittmann presented students with a Gaussian pulseshape and asked them questions which probed their understanding of and ability to describe the wave motion. Wittmann found that many students revised their physical understanding to fit their misinterpretations of the mathematics or vice versa. “Students often used misinterpretations of the mathematics to guide their reasoning in physics or they used misinterpretations of the physics to guide their understanding of the mathematics” (pp. 56). Wittmann suggests that more research needs to be done in this area to more deeply investigate student understanding.

Wittmann’s study suggests that a link exists between students’ understanding of mathematical and physical concepts. Furthermore, Wittmann has shown that students are willing to revise their understanding of a mathematical concept in order to fit with a physical misinterpretation or modify a physical understanding in order to fit with a mathematical misinterpretation.
Wittmann’s study highlights ways in which students use their misunderstandings of mathematics to influence their conceptualizations of physics concepts. His study is an important link between the mathematics and physics education literatures. In order to gain a deeper understanding of how students conceptualize the relationship between mathematics and physics concepts, more studies of this nature need to be undertaken. Whittmann’s study begins to investigates how students use mathematics to aid in their conceptualizations of physics concepts. Further investigations are needed to examine how students use their understanding of physics to influence their conceptualizations of mathematics concepts. The present study attempts to begin to address the latter issue.

Summary

The research presented in this chapter complements the theoretical framework discussed in Chapter II. Many of the research studies presented in this review employed qualitative, descriptive research methods, based on a constructivist epistemology. A number of the studies considered in this literature review report the results of teaching experiments, case studies, and the effects of reform curricula on students’ understanding of calculus concepts. The results of many of the research studies presented are used to inform calculus and physics curriculum development.

The review of the literature suggests several considerations when investigating students’ conceptualizations of calculus concepts. Foremost, the reviews point to a need for theory development in the area of student conceptualization of calculus concepts, especially investigations relating students’ physics experiences to calculus concept development (Ferrini-Mundy & Graham, 1991). Additionally, research should
investigate students' capacity to work with calculus concepts in multiple contexts, not simply their ability to apply formulas.

Several researchers stressed the importance of considering students' prior, informal experiences with mathematics concepts (Speiser & Walter, 1996; Nemirovsky, Wright & Tierney, 1998). Students' experiences shape how and what they learn in the mathematics classroom. Students' experiences also need to be considered when investigating students' conceptualizations of calculus concepts.

Finally, many results concerning students' graphical misconceptions were replicated in both the calculus and physics literatures. Trowbridge and McDermott (1989) and Goldberg and Anderson (1984) found that students often confuse position and velocity criteria. In particular, students use height, rather than slope to answer speed comparison questions presented in a graphical context. Clements (1989) reported similar findings in an independent mathematics education investigation.

Nemirovsky and Rubin (1992) reported on students' tendency to draw the graph of a function that closely mimicked the shape of the derivative graph. For example, if a student encountered a linear, increasing velocity graph, he or she tended to draw a linear, increasing position graph. McDermott, Rosenquist, and van Zee (1987) reported similar findings in an earlier, independent study. The investigators of these studies made recommendations for curriculum development and teaching practice based on the outcomes of their studies.

The literature review, along with the theoretical framework, serves to inform the research questions, the types of data collected, and the methodology employed in the present research study. Primarily, the literature review situates the present study within
the assemblage of existing research studies. The literature review also identifies issues and problems in need of further investigation. Finally, the literature review serves to affirm theoretical assumptions underlying the present research study.
CHAPTER IV

METHODOLOGY

This research study explored how students use physics concepts to inform their conceptualization of calculus concepts. The main research question that the present study addressed is: How do students draw upon physics concepts to inform their understanding of average rate of change, derivative, and integral? Secondary research questions were posed to investigate if students' misunderstandings of physics concepts misinforms their understanding of calculus concepts; if students consistently use physics in a certain way to help them understand calculus concepts; and to describe students' concept images of average rate of change, derivative, and integral. Tall and Vinner's (1981) notion of concept image, a constructivist theory of learning, and a definition of representation helped shape the research questions and set the stage for the methodology. As stated previously, qualitative research methods of data collection and analysis were chosen to investigate the research questions because qualitative research methods fit best with the theories and assumptions that constitute the framework for the present study.

The presentation in this chapter will begin with an overview of the study, followed by a detailed discussion of the research design, procedure, instruments, and analysis techniques.
Overview

The present research study utilized a multiple case study design with analysis by and across cases. The cases represent eight first year students in the College of Engineering and Physical Sciences at the University of New Hampshire who were enrolled in an integrated Calculus/Physics program. The research plan consisted of three main data gathering parts: (1) conducting semi-structured task-based interviews, (2) participant-observation in the Calculus/Physics course, and (3) obtaining copies of students' in-class notes, in-class activities, homework assignments, and examinations. The data was gathered in order to solicit information about how the eight students were using physics as they worked through calculus problems. The interview tasks were designed to elicit information about how the students used physics to help them solve calculus problems presented in various contexts. The classroom observations focused on the language used by both the instructor and the Calculus/Physics students. During the classroom observations, I paid particular attention to the eight students participating in the present research study. Finally, the students' work was collected to gather more evidence about how the students were using physics to help them solve calculus problems. The students' work was used in the data analysis primarily for triangulation and verification purposes.

The rationale behind the research plan was threefold: (1) To examine the manner in which students use physics to aid in their construction of calculus concepts, (2) To carefully examine the mathematical constructs that the students formed from their participation in an integrated Calculus/Physics program, and (3) To develop a detailed portrait of each students' concept image of derivative, integral, and rate of change.
description of each students' concept image was developed by analyzing the students' responses to interview tasks and triangulated with student-produced concept maps, observations of students in class, and students' homework, performance on examinations, and class work. A second layer of analysis resulted in the emergence of a classification scheme that describes how the students use physics to inform their conceptualization of calculus concepts. Finally, by searching individual student descriptions for patterns and similarities, a general description for the interactions between concept image and classification was proposed.

Research Design

The constructivist theory of learning has had a profound impact on mathematics education research. Paul Ernest (1998) claims that the widespread acceptance of constructivism as a learning theory in the domain of mathematics education has greatly contributed to the shift toward more qualitative research in the past few decades. Furthermore, Ernest (1998) claims that constructivism has led to a set of new research emphases central to qualitative research. In particular, qualitative research attends to previous constructions that learners bring with them; the social contexts of learning; the beliefs and conceptions of knowledge of the learner, teacher, and researcher (Ernest, 1998, pp. 31). Ernest defines qualitative research as, “primarily concerned with human understanding, interpretation, intersubjectivity, [and] lived truth” (pp. 33). The aim of qualitative research is to explore the details of a particular phenomenon and analyze those features of the phenomenon that may serve as an example of something more general. In addition to being a natural consequence of assuming a constructivist epistemology as
described by Ernest above, qualitative research methods were selected for this study because I wanted to generate data rich in detail and embedded in the context of an interdisciplinary Calculus/Physics class. Using what Geertz (1973) calls "thick description", I set out to articulate how the experiences of the students, both in class and outside of class, inform their understanding of calculus concepts.

I primarily used case study techniques for the data collection and analysis. Stake (1995) describes case study as, "The study of the particularity and complexity of a single case, coming to understand its activity within important circumstances" (pp. xi). The goal of case study research is to understand the complexity of a single case, a case being a person, group of people, an event, or a program. As Stake (1995) notes, "The case is a specific, a complex, functioning thing" (pp. 2).

One of the characteristics of case study research that distinguishes it from other qualitative research traditions is the bounded focus of the case. Saying that a case is bounded means that the case is an object or system, rather than a process and that time and place bound the case. For example, a teacher is a case, but a teacher's teaching lacks the boundedness to be considered a case (Stake, 1995, pp. 2). The cases in this study, that is the students, are bounded both by time and place. The time interval in which I am studying the cases is the two-semester duration of their involvement in the Calculus/Physics program. The setting of the Calculus/Physics class also bounds the case since it is a finite, physical place.

The choice of case study design was also informed by the theoretical framework, specifically, the constructivist theory of learning. As Noddings (1990) noted, assuming a constructivist theory of learning implies the adoption of a constructivist methodology.
The constructivist methodology is concerned with understanding individuals’ behaviors by investigating their reasoning, purposes, and perspectives. Additionally, the constructivist learning theory used to frame the present research study is based on the work and theory of Jean Piaget. Piaget primarily used clinical interviews as a source of data collection in his work. Clinical interviews are a major source of data in case studies. Finally, Stake (1995) demonstrates the dependence of the case study design on a constructivist viewpoint:

Case study research shares the burden of clarifying descriptions and sophisticating interpretations. [A] constructivist view encourages providing readers with good raw material for their own generalizing. The emphasis is on description of things that readers ordinarily pay attention to, particularly places, events, and people, not only commonplace description, but ‘thick description,’ the interpretations of the people most knowledgeable about the case. Constructivism helps a case study researcher justify lots of narrative description in the final report (pp. 102).

The ‘raw material’ used in the present research study was obtained from clinical interviews with students, classroom participant-observation, and collection of student work.

Creswell (1998) contends that “The [case study] researcher needs to have a wide array of information about the case to provide an in-depth picture of it” (pp. 39). For each student, I collected information from clinical interviews with students, classroom participant-observation, and collection of student work in order to paint an in-depth picture for the reader. With the collected data, I constructed a picture of each student’s conceptualizations of calculus concept through the themes of representation and physics use. My data collection process is consistent with Creswell’s (1998) call for data collection to draw upon multiple sources of information.
The analysis of the data employs one or more of a variety of qualitative strategies for data analysis. Creswell (1998) maintains that a typical format for the analysis of data from multiple cases first involves a within-case analysis followed by a cross-case analysis. A within-case analysis occurs when the researcher identifies themes within a single case. For multiple case studies, this analysis may suggest themes unique to a case or themes common to all cases studied (Creswell, 1998, pp. 252). I will talk more about the specifics of my within-case and cross-case analyses in a forthcoming section.

I also used some techniques from grounded theory to aid in the analysis of the data. Strauss and Corbin (1998) define grounded theory as, “Theory that was derived from the data, systematically gathered and analyzed through the research process” (pp. 12). Techniques of analysis from grounded theory were used since it was my expectation that categories would emerge from my data, even though I could not conceptualize these categories a priori. Strauss and Corbin (1998) assert that “Theory derived from the data is more likely to resemble ‘reality’ than theory derived by putting together a series of concepts based on experience or solely through speculation” (pp. 12).

The grounded theory technique used for data analysis is microanalysis. Microanalysis is a detailed line-by-line analysis frequently conducted at the beginning of a study in order to generate initial categories (Strauss & Corbin, 1998, pp. 57). Strauss and Corbin (1998) discuss several functions of employing microanalytic techniques to analyzing data. I will restate a few of them here:

1. Microanalysis obliges the researcher to examine the specifics of the data. Hence, the focused nature of this analysis allows the researcher to break the data apart and reconstruct them according to interpretive categories.
2. Microanalysis compels the researcher to listen closely to what the subjects are saying. One goal of microanalysis is to understand how the subjects are interpreting and making sense of events. In closely listening to the subjects' words, the researcher is forced to consider alternative explanations and refrain from initially laying his/her interpretation on the data.

3. Microanalysis is considered a theoretical coding approach since in conducting microanalysis, the researcher attempts to conceptualize and classify events. Classification means grouping events, actions, and outcomes according to similarities and differences. Theoretical coding differs from descriptive coding in that the outcome of theoretical coding is the emergence of a classification scheme, whereas the outcome of descriptive coding is simply describing a setting or event.

Strauss and Corbin (1998) assert that microanalysis can occur at any point in the analysis of the data, but it is a necessary first step in analysis of one's data. The authors also claim that microanalysis also can be used to revisit old data or make sense of puzzling pieces of data. I will discuss the microanalysis of data in this research study in a forthcoming section.

Procedure

Setting

The research took place at the University of New Hampshire, where an interdisciplinary Calculus/Physics class is being offered to first-year students as an alternative to enrolling in separate calculus and physics classes. The University of New Hampshire College of Engineering and Physical Sciences (CEPS) requires that most of
its students take two semesters of calculus (differential and integral), one semester of
differential equations, and two semesters of physics. The differential and integral
calculus classes, as well as the physics classes are prerequisites for many of the upper-
level classes in the College of Engineering and Physical Sciences.

In 1998, the University of New Hampshire received a grant from the National
Science Foundation to develop, implement, evaluate, and disseminate information about
an interdisciplinary calculus and physics program for first year science, mathematics, and
engineering students. The program was developed during the spring and summer of
1998. The departments of mathematics and physics began offering the interdisciplinary
Calculus/Physics class to CEPS students in the fall of 1998. This two-semester sequence
satisfies the General Physics I and II requirements and the Calculus I and II requirements.

The Calculus/Physics course covers roughly the same material as the General
Physics I and II and the Calculus I and II classes. One major difference between the
Calculus/Physics course and the standard introductory courses in physics and calculus is
that the Calculus/Physics curriculum was developed around two overarching themes:
change and superposition. The idea of change, how one describes and works with values
that are constantly changing, helps guide the organization of topics during both semesters
of the course. The notion of superposition, the idea that we can understand complex
phenomena by breaking it down into smaller, simpler pieces and then adding the effect of
the small pieces to get the whole effect, helps guide the organization of topics during the
second semester.

The format of the Calculus/Physics course is based on the Studio model pioneered
at Rensselaer Polytechnic Institute. Each class is a mixture of short lecture, group
activities, computer work, and experiments. The class meets five days a week for two hours a day. Typically, two days are devoted to calculus topics and taught by a calculus instructor and two days are devoted to physics instruction and taught by a physics instructor. Class on the fifth day features both instructors and focuses on connections between calculus and physics and problem solving. Copies of the topic schedules for the course are included in Appendix B.

The instructors lay out two main goals for the course in the Calculus/Physics course syllabus: (1) For students to improve their ability to understand and use the concepts of change and superposition, and (2) For students to improve their ability to solve complex, real-world problems. Additionally, the following secondary goals are also stated in the syllabus as follows:

In addition to the main goals, we have several secondary goals. At the end of the school year you should have significantly improved your ability to:

- Carry out essential operations
- Reason logically and defend your ideas
- Learn on your own
- Work in groups
- Apply physics and calculus concepts to a wide range of situations

The students are expected to spend at least ten hours per week outside of class on the calculus/physics course. The instructors frequently stress the importance of class attendance and have instituted a policy such that for each class a student misses without a legitimate excuse, one half of a point will be deducted from his/her final grade, up to five points. Typically, students enrolled in the Calculus/Physics program miss relatively few, if any classes during the year.

The textbook used in the calculus portion of the class is *Calculus of a Single Variable: Early Transcendental Functions* by Larson, Hostetler, & Edwards (1999). The

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textbook is used largely as a resource for the students and for the instructor to assign homework problems. In addition, the students use daily calculus activities that were created for the course by Dr. Kelly Black, Associate Professor of Mathematics at the University of New Hampshire and others at the University of New Hampshire. The activities were designed for students to explore calculus ideas and allow for students to make connections between calculus and physics.

The textbook used in the physics portion of the class is *Fundamentals of Physics* by Halliday, Resnick, and Walker (2000). Again, the textbook is used mainly as a resource for students and for the instructors to assign homework problems. In addition, the students work from the book *Tutorials in Physics*, by McDermott and her colleagues and activities created by Dr. Dawn Meredith, Associate Professor of Physics at the University of New Hampshire.

The ordering of the calculus and physics topics contributes greatly to the integrated curriculum. The curriculum is designed for the students to see the applicability of the calculus as they learn it, and conversely that they have all the mathematics they need to solve the current physics problems. In order to coordinate the calculus and physics topics in the class, the presentation of calculus topics is reordered. The four basic threads of calculus (function, continuity, derivative, and integral) are discussed first for polynomial functions only and then again for the other classes of functions (logarithmic/exponential and trigonometric) as they arise in the physics curriculum.

This reordering of the calculus curriculum allows for the presentation of the physics and calculus content in a more unified way and gives the mathematics a rich
context. For example, by the end of the first month of the class, students can use antiderivatives to calculate velocity and position as a function of time from an acceleration equation. In contrast, students enrolled in the traditional physics class at the University of New Hampshire spend a good deal of time learning algebraic manipulations of the constant acceleration equations and often fail to understand that these equations are limited in their applicability.

The topic schedules in Appendix B show the day-to-day arrangement of the calculus and physics topics for the 2000-2001 academic year. Consider the arrangement of the topics for September 11 and September 12 on the fall 2000 Calculus/Physics schedule. On September 11, 2000, the students attended class in the physics laboratory and worked on an activity from the *Tutorials in Physics* book by McDermott, Shaffer, et. al. (1998). The activity involved the students predicting graphs of velocity and acceleration given a graph of position, and vice versa. The students checked their predictions by creating the motion with a cart on a track and a motion detector. The mini-lecture in class on September 11 involved a discussion about average velocity and average acceleration. The physics instructor prompted the students to begin thinking about the meanings of instantaneous velocity and instantaneous acceleration by having them consider what happens to the velocity and acceleration as time intervals become smaller and smaller. The physics instructor indicates that the students will continue the conversation about instantaneous velocity and acceleration the next day in calculus class.

In calculus class on September 12, 2000, the calculus instructor revisits the discussion that the students and the physics instructor took part in during the previous day’s physics class. The calculus instructor, however, begins to push the students to
think about abstract notions of average and instantaneous rates of change by considering a graph of an arbitrary function, \( f(t) \). The class engages in a conversation about the average rate of change of \( f(t) \), similar to the discussions about average and instantaneous velocity and acceleration the previous day. The students work through activities in the calculus class that involve both average and instantaneous velocity and average and instantaneous rates of change for arbitrary functions. This example illustrates how the calculus and physics curriculums are integrated in the Calculus/Physics program.

Although the Calculus/Physics program integrates the calculus and physics curriculum into a single, unified curriculum, as described above, the students receive separate grades for calculus and physics. The students’ final grades in both calculus and physics are based on class attendance and participation, homework, group projects, and examinations. Students are generally assigned one calculus and one physics homework set each week. During the 2000-2001 academic year, the physics instructors initiated the use of WebAssign, a web-based homework system. WebAssign grades the students’ homework assignments and gives immediate feedback to the students. Finally, the students take three tests during each semester and a final examination at the conclusion of each semester.

The interdisciplinary calculus and physics class is offered to approximately 50 students each year (two sections of the course with 25 students in each section). All physics and electrical engineering majors are invited to participate in the program since they are the only CEPS students who are required to enroll in introductory physics and calculus concurrently during their first year at the university. Students majoring in other CEPS disciplines (mathematics, chemical engineering, etc.) are invited to participate in
the class based on their major, background in high school calculus and physics, and participation in the University of New Hampshire honors program. The majority of the students in the interdisciplinary Calculus/Physics program are engineering majors. About half of the students in the Calculus/Physics program are enrolled in the honors program.

Students enrolled in the interdisciplinary Calculus/Physics program are exposed to a curriculum that is based on the connections between calculus and physics. Thus, the Calculus/Physics students explicitly see connections between calculus and physics topics. The Calculus/Physics students are learning calculus concepts in a context that depends on their understanding of physics. The students in the Calculus/Physics program have physical interpretations of calculus concepts readily available to them. Studying these students will allow me to examine how the context of an interdisciplinary class affects their understanding of calculus concepts. Additionally, previous investigations of students’ understanding of calculus concepts have focused on students enrolled in non-integrated classes. Researchers have identified a need for research studies that investigate the role of physics concepts and examples in students’ understanding of calculus concepts (Ferrini-Mundy & Graham, 1991). The present research study differs from previous research on students’ understanding of calculus concepts in that it is set within the context of an interdisciplinary class.

Subjects

Eight students participated in the present study during the 2000-2001 academic year. The eight students were enrolled in the Calculus/Physics course for two semesters. Students were selected to participate in this study based on the information generated by the Average Rate of Change Pretest (see Appendix A). The Average Rate of Change
Pretest was designed to obtain background information about students’ experiences with the concept of rate of change and to measure students’ abilities to solve average rate of change problems. The Average Rate of Change Pretest will be discussed in more detail in a forthcoming section. Students were chosen to participate in this study based on their reported backgrounds in both mathematics and physics and their familiarity with the concept of rate of change. I intended to select a cross-section of students with varying calculus backgrounds and experience with the concept of rate of change to participate in this study. My goal was to create a sample of students whose range in abilities span the abilities represented in the Calculus/Physics class, thus allowing me to check for themes in my data across student ability groups and to contrast themes between student ability groups as part of the data analysis. Thirty-seven out of 51 students were contacted and asked to participate in clinical interviews. The clinical interviews transpired throughout the two-semester sequence of the class.

Out of the 37 student originally contacted, twelve students responded that they were willing to take part in my study. Three students dropped out of the study after the first interview and one student dropped out of the study after the second interview. The eight remaining students represent the core group for this study. They completed two additional interviews and furnished me with copies of their examinations, homework, and class notes from the first semester of the course, Fall 2000.

The eight subjects in this study consisted of seven males and one female student. The gender balance in the present study is reflective of the gender balance in the Calculus/Physics class. During the fall 2000 semester the Calculus/Physics class enrolled 42 males and 8 females. The majors of the eight subjects in the present research study
are shown in Figure 4. The majors of the students enrolled in the Calculus/Physics program during the Fall 2000 semester are shown in Figure 5.

Figure 4: Majors of Subjects in Present Study

Figure 5: Majors of Students Enrolled in the Calculus/Physics Class, Fall 2001
I believe that the eight students in the present study adequately represent the students enrolled in the Calculus/Physics class during the Fall 2000 semester. Notice that approximately half (44%) of the students in the Calculus/Physics class were Electrical or Mechanical Engineers. Half of the students in the present study are Electrical or Mechanical Engineers. The number of students in the Calculus/Physics class (20%) and the number of students participating in the present study (25%) who had undeclared majors in the College of Engineering and Physical Science were also very similar. Since the Calculus/Physics class enrolled so few Chemical Engineering, Civil Engineering, and Computer Science majors (4% each), I felt that it was not necessary to concentrate on representing these groups in the eight students.

Five out of the eight subjects in the present research study were enrolled in the University of New Hampshire honors program. Twenty-five out of the fifty students enrolled in the Calculus/Physics program were also enrolled in the University of New Hampshire honors program.

Duration

I collected data over the two-semester duration of the interdisciplinary Calculus/Physics course during the 2000-2001 academic year. Since the calculus topics I was interested in studying were introduced throughout two semesters, I collected data during both the fall and spring semesters. However, due to practical constraints, most of the data collection focused on the first semester of the course. I conducted three interviews with each of the eight students during the first semester and one interview with each student during the second semester. I attended many of the calculus sessions of the
class and a smaller portion of the physics sessions and combined Calculus/Physics sessions.

Data Collection

Part I: Semi-Structured Task-Based Interviews

The primary source of data collection is semi-structured, task-based interviews with the student participants. The interviews were planned around a series of tasks. Some tasks were designed solely by myself and others were adapted from the literature. The interviews were intended to probe students’ understanding of calculus concepts with specific emphasis on determining the students’ concept image of rate of change, derivative, and integral and examining how the students used physics to help them solve the calculus tasks. Each interview lasted approximately one hour and was audiotaped with the consent of the interviewees. Semi-structured task-based interviewing was chosen because this technique allowed me to probe and question students’ understanding of calculus concepts in a detailed manner. Goldin (2000) describes structured task-based interviews in this way:

Structured task-based interviews for the study of mathematical behavior involve minimally a subject (the problem solver) and an interviewer (the clinician), interacting in relation to one or more tasks (questions, problems, or activities) introduced to the subject by the clinician in a preplanned way. ...Explicit provision is made too for contingencies that may occur as the interview proceeds, possibly by means of branching sequences of heuristic questions, hints, related problems in sequence, retrospective questions, or other interventions by the clinician (pp. 519).

Goldin goes on to distinguish a structured interview from an unstructured interview in the following manner:

It is this explicit provision for contingencies, together with the attention to the sequence and structures of the tasks, that distinguishes the ‘structured’
interviews discussed here from the 'unstructured' interviews, which may be limited to 'free' problem solving (where no substantial assistance that would facilitate a solution is given by the clinician to the subject) or the handling of contingencies on an improvisational basis (pp. 519).

My interviews were *structured* task-based in the following way, according to Goldin’s definition: the tasks were introduced to the subjects in a preplanned way. Furthermore, I prepared branching and retrospective questions for some of the tasks. The branching and retrospective questions were intended not to lead the students to an answer, but to elicit information about the students’ thought processes.

My interviews were *unstructured* task-based in the following way: I did not give substantial assistance to the students, in particular the type of assistance that would lead to their solution of a problem. Furthermore, I handled some contingencies on an improvisational basis. Thus, my interviews were *semi-structured* task-based, since I blended attributes of both the structured and unstructured interviews into my protocol.

Each student was trained in the think-aloud protocol technique before the first interview. In the training, I described the think-aloud protocol to the students and then asked them to complete a non-mathematical task and a series of mathematical tasks using the protocol. During the think-aloud protocol training period, I offered the students feedback on their use of the protocol.

My role in the interviews was strictly that of a clinician. I refrained from giving the students feedback on their work during the interviews, although I offered to discuss any of the problems with the students after the completion of the interview. I asked the students to clarify their procedures when I felt that I did not completely understand their reasoning or if I believed that I could be (mis)interpreting their reasoning. This required not only attention to the students’ verbalizations but also monitoring my own inferences.
during the interviews. I informed the students that I was not judging their work in any way and reiterated throughout the interviews that I was interested in how they were solving the problems, not whether or not they were able to solve the problem correctly. It was my intention to create a non-threatening atmosphere by telling the students that the focus of the interviews was to ascertain how they solved the tasks, not whether they solved the tasks correctly. Although I was interested in the students’ overall performance on the task, I felt it necessary to downplay the importance of a correct answer so that the students would feel comfortable talking aloud about their solutions, even if they produced incorrect answers.

Part II: Classroom Participant- Observation

Acting as a participant-observer in the class allowed me to collect data about the context in which students learn the formal calculus concepts. Data collected from participant-observation allows me to offer a detailed description of the students’ learning environment. Additionally, participant-observation data allows me to provide the reader a context in which to view the results of the present study. Finally, other researchers who wish to determine transferability of the results of this study can use the description generated by my participant-observation. By comparing the environment of the Calculus/Physics class to another setting, researchers may determine the transferability of particular results of the present study.

Wolcott (1995) proposed three questions for researchers to ask themselves in order to ascertain if they will be able to observe or experience the phenomena they are interested in observing and experiencing in the field. These questions are:

1. Can whatever I want to study be ‘seen’ by a participant-observer at all?
2. Am I well positioned to observe the phenomena?

3. What are my own capabilities for participating and observing in this situation?

I will talk briefly about my background and involvement with the Calculus/Physics program in order to address Wolcott's concerns.

As previously mentioned, part of my involvement in the Calculus/Physics program entailed helping develop a set of in-class activities to be used in the calculus portion of the class. I also was employed as a teaching assistant in the program for two years. My duties as a teaching assistant involved attending and assisting in sections of the class. During class, I answered students' questions and guided them as they worked through the in-class activities. In addition, I held office hours outside of the scheduled class time and acted as an advisor for student projects. During the first year of the program (1998-1999 academic year), I assisted in both the physics and calculus sections of the class; thus, I attended every class meeting of the Calculus/Physics program that year. My attendance in both the calculus and physics sections of the class allowed me to see how the class ran as an integrated program from the perspective of the students.

In addition to my duties as a teaching assistant, I also designed and implemented an evaluation of the Calculus/Physics program. The aim of this evaluation was to determine if the program was meeting its goals. The goals of the Calculus/Physics program include improving students' problem solving skills, improving students' understanding of the conceptual foundations of the operations and processes essential to calculus and physics, and enriching students' awareness of the connections between calculus and physics. Many of the evaluation instruments were comparative in nature; that is, I strove to distinguish similarities and differences between students enrolled in the...
Calculus/Physics program and those students enrolled in the typical calculus and physics classes at the University of New Hampshire. The results of this evaluation have been synthesized and discussed elsewhere (see Marrongelle, Meredith, & Black, in press).

My experiences as a teaching assistant and evaluator of the Calculus/Physics program provided me with a deep understanding of the character and effectiveness of the class. These experiences gave me insight into the class and positioned me to focus on the eight students taking part in the present study. Because I was familiar with the day-to-day details of the class, my participant-observation focused on the eight students. I was able to draw on my prior experiences in the class to fill in details that otherwise might not have been observed.

While I was a participant-observer, I interacted with the Calculus/Physics students on various levels during each class session that I observed. In the calculus session, I sat with different groups of students at their computer pods during the mini-lectures. During the mini-lectures, I took notes on the language of both the instructor and the students and paid close attention to their use of calculus and physics concepts and terminology. While the students worked on activities in their groups, I rotated around the room, taking notes on the students’ conversations. Again, I paid particular attention to the eight students’ use of physics concepts and terminology as they solved calculus problems. Occasionally the students would ask me questions about the problems they were working on, or would ask me to check their answers to the problems. When the students asked me questions, I gave them limited feedback and prompted them to check their answers with their group members. Generally, the students would engage in discussions with their group
members. I took notes on these discussions, again focusing on the students' uses of physics in their conversations with each other.

Part III: Student Work

I collected and photocopied the eight students' in-class activity sheets, class notes from the calculus and physics sections, examinations, and homework assignments. I collected the students' in-class activity sheets, homework assignments, and examinations in order to help ascertain how the students worked with calculus concepts presented in various contexts (physical, graphical, numeric, symbolic) and to look for places where the students exhibited misconceptions of physics concepts and how those misconceptions affected their conceptualizations of calculus concepts. I collected the students' physics and calculus class notes in order to check my own classroom observation notes and also to look for instances where the students made connections between physics and calculus. These data allowed me to clarify and support findings from the clinical interviews and in-class observations.

Instruments

Many of the research instruments used in the present study were pilot-tested during different phases of the overall evaluation of the Calculus/Physics program. For example, the Average Rate of Change Pretest was piloted with students enrolled in the Calculus/Physics program during the 1999/2000 academic year. Furthermore, most of the interview tasks were developed, tested, and refined during clinical interviews conducted with a cohort of students involved in the 1999/2000 phase of the
Calculus/Physics evaluation. I will begin my discussion of the research instruments with a description of the Average Rate of Change Pretest.

**Placement Instrument: Average Rate of Change Pretest**

During the first week of class in the fall 2000 semester, all students enrolled in the Calculus/Physics class completed a pretest designed to probe their knowledge of average rate of change. The Average Rate of Change Pretest was designed to obtain background information about students' experience with the concept of rate of change and to measure students' abilities to solve average rate of change problems. (See Appendix A for a copy of the Rate of Change Pretest.) Questions and problems on the Rate of Change Pretest were adapted from the work of other researchers (Orton, 1983) or solely developed by myself.

The first question on the pretest asked students to place themselves in a category that most appropriately represented their experience with the rate of change concept in high school. The students placed themselves in one of the following categories: (1) No previous experience with the definition of rate of change; (2) Experience with an informal definition of rate of change; (3) Experience with the formal definition of rate of change. If a student checked either the second or third category, he or she was asked to define rate of change. If a student placed him/herself in the first category, he or she was asked to provide a definition of rate of change.

Other questions on the Average Rate of Change Pretest were designed to gather information about students' abilities to work with rate of change in symbolic, graphic, numeric, and physical contexts. Data were presented to the students in one or a combination of the above-mentioned contexts. Students were given approximately 20 to 25 minutes to work on the Average Rate of Change Pretest on the first day of class.
I classified students based on their experience with the concept of rate of change and their ability to solve a rate of change graph problem. The students were classified according to their self-reported information on the Average Rate of Change Pretest. The first question used for screening purposes asked the students what their experience with the concept of rate of change had been. Students placed themselves in one of three categories and then answered a follow up question based on the category the student placed him/herself into. Since this information was self-reported, I felt it necessary to follow up on the self-reported information with a question on the Average Rate of Change Pretest that checked the students' ability to work with the concept of average rate of change in a problem.

The second question I used to screen the students was a four-part problem in which the students were asked to find the average rate of change between different sets of points on the graph of \( f(x) = x^2 \). Two of the questions asked the students to compute an average rate of change that yielded an integer answer. One question yielded an answer of zero. The final question asked the students for a general formula for the average rate of change between any two points on the graph of the function.

In order to categorize students based on their answers to this four-part question, I broke the students into two groups: those who answered at least 3 out of 4 of the parts correctly and those who answered less than 3 out of 4 of the parts correctly.

In order to compare the backgrounds of the eight students in the present study with the backgrounds of the students in the Calculus/Physics class, refer to the following two charts. The breakdown of the eight students' familiarity and ability to work with the concept of rate of change is shown in Table 2.
The breakdown of the Calculus/Physics class’s familiarity and ability to work with the concept of rate of change is shown in Table 3.

<table>
<thead>
<tr>
<th></th>
<th>GREATER THAN OR EQUAL TO ¾ CORRECT</th>
<th>LESS THAN ¾ CORRECT</th>
<th>TOTAL</th>
</tr>
</thead>
<tbody>
<tr>
<td>NO PRIOR EXPERIENCE</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>INFORMAL EXPERIENCE</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>FORMAL EXPERIENCE</td>
<td>3</td>
<td>4</td>
<td>7</td>
</tr>
<tr>
<td>TOTAL</td>
<td>3</td>
<td>5</td>
<td>8</td>
</tr>
</tbody>
</table>

Table 2: Rate of Change Pretest Selected Results for Subjects in Present Study

Most of the subjects in the current study (87.5 %) indicated that they had experience with the formal definition of average rate of change in the past. 53 % of the Calculus/Physics students placed reported that they had experience with the formal definition of average rate of change. More students in the Calculus/Physics class (55%) answered 75% or more of the four-part question correctly than the subjects participating in the current study (37.5%). Interview Tasks

**Background Questions.** I collected information about each student’s academic and non-academic background for purposes of describing the students. I collected
information about each student's high school mathematics and physics classes during the first interview. Each student orally responded to a set of question about their prior high school experiences. (The background questions are listed in Appendix A.) Additional background information, such as major, involvement in extracurricular activities, and reasons for enrolling in the Calculus/Physics class was collected via an e-mail survey during the Spring semester. (See Appendix C for a copy of the survey.)

**Problems.** The interview tasks consisted of a series of problems related to the concepts of rate of change, derivative, integral, and the Fundamental Theorem of Calculus. The tasks were either developed by myself or adapted from the work of others. Some of the tasks closely resembled problems the students had worked on in the calculus classroom activity book. Other tasks consisted of problem situations that the students had not encountered in the Calculus/Physics course and were therefore unfamiliar to the eight students.

The purpose for selecting a range of familiar and unfamiliar tasks was twofold: (1) To ascertain the students' understanding of concepts based on familiar problem contexts and (2) To determine if the students could extend their understanding to new problem contexts. See Appendix A for copies of the tasks and follow-up questions presented to the students during the interview sessions.

**The First Interview.** The first interview focused mainly on the students' responses to the Rate of Change pretest and took place within two weeks of the first day of class. During the first two weeks of class, the students were introduced to the concepts of average rate of change, instantaneous rate of change, sequences and convergence, and derivatives of polynomials in the calculus sessions and average velocity, the relationship
between position, velocity, and acceleration, vector addition, and motion on an inclined plane during the physics sessions. I presented each student with a copy of his/her Average Rate of Change pretest and asked the student to elaborate on his/her answers. In some instances, students had not completed a problem on the Average Rate of Change Pretest so I asked the student to solve the problem using the think-aloud protocol. An additional task, in which the student was presented with a graph of position versus time and asked to estimate the average rate of change from the graph was also given to the students during the first interview. I attempted to establish the students’ familiarity working with graphical contexts of the derivative by asking the students questions concerning where the graph of the velocity would be positive and negative.

The Second Interview. The second set of interviews took place in mid-October, one to two weeks after the students had taken their first examination in the Calculus/Physics course. In the two weeks that I was conducting the second set of interviews, the students were learning about the various techniques for differentiation (product rule, quotient rule, chain rule), inverse functions, including exponential and logarithmic functions, and Newtons’ Second Law. The tasks in the second interview focused mainly on students’ ability to work with graphical contexts of the derivative.

The students were presented with a series of four graphs of functions and were asked to produce the graphs of the first derivative, second derivative, and/or the antiderivative. Some of the graphs were presented to the students as abstract mathematical functions and other graphs were given in the context of kinematics.

The students were also given a task adapted from Bezuidenhout (1998) which was designed to answer the question: How useful and operational are students’ concept
images of rate of change? (Bezuidenhout, 1998, pp. 395). Bezuidenhout concluded that many of the students in her study possessed concept images that were deficient with respect to the graphical aspects of rate of change. The results of the eight students in the present study on the task adapted from Bezuidenhout will be discussed in Chapter VI.

The Third Interview. The third set of interviews took place in the beginning of November, as the students were using the Fundamental Theorem of Calculus to examine conservation of momentum and explore work integrals. The tasks for the third interview focused on students’ understanding of the Fundamental Theorem of Calculus and the difference between antiderivative and integral. Some of these tasks were based on the in-class and homework problems that the students worked on in the Calculus/Physics class. Another task was adapted from Ferrini-Mundy and Graham (1994) to probe students’ understanding of the difference between anti-derivative and integral. Other tasks were designed to probe students’ understanding of the Fundamental Theorem of Calculus.

The Final Interview. Finally, I interviewed each student once during the spring semester, 2001. This last interview focused on students’ ability to work with data presented in numeric and physical contexts. Tasks were adapted from the Hughes-Hallet, Gleason, et. al. (1994) and Ostebee and Zorn (1992) calculus textbooks. One task asked students to estimate the integral of a function from a function table. This task was designed to probe students’ understanding of the integral as area under a curve. Another task, which asked the students to sketch the graphs of position, velocity, and acceleration of a spring-mass system, was designed to probe the students’ ability to work with derivatives and integrals in a physical context. Other tasks were designed to gauge students’ conceptualization of rate of change.
Summary of Problems. The interview tasks were designed by myself or adapted from the work of others so that I might probe the students’ conceptualizations of rate of change, derivative, and integral. I also used students’ homework assignments and examinations to judge the students’ ability to work with calculus problems presented in multiple contexts. However, the interview tasks and homework problems only told one part of the story, namely, how well a student could apply knowledge to specific problem situations. Another part of the story is how each student mentally organizes concepts. I turned to concept maps to help me understand the students’ mental organization of concepts.

Concept Maps. Concept maps are a tool used by teachers and researchers for a myriad of purposes: to promote student reflection, and to serve as an organizational method for aiding understanding of new subject matter, to measure change over time, or to aid in understanding how students think about a certain concept. Novak and Gowan (1984) present concept maps as, “a way to visualize concepts and the hierarchical relationships between them” (pp. 28). Concept maps are viewed as a graphical externalization of the organization of a student’s knowledge within a particular domain.

Originally conceived of as a tool to represent conceptual changes in students over time (Novak, 1990), concept maps have evolved as a means to assess cognitive structure at a specific time (Laturino, 1994; Williams, 1998; Harnisch, Sato, Zheng, Yamagi, & Connell, 1994), and as instructional aides within the classroom (Novak, Gowin & Johansen, 1983; Okebukola, 1990). More recently concept maps have become popular in teacher education research (Beyerbach & Smith, 1990; Portnoy, Graham, Berk, Guttman,
& Rusch, 1998) to measure change in teachers’ beliefs and attitudes towards mathematics.

Researchers have used concept maps to assess students’ cognitive structures in a number of different ways. Some researchers give students a list of terms (vocabulary list) and have students draw concept maps using the given vocabulary. A researcher using concept maps in this way could focus on how a student is connecting together specific ideas related to a concept. Other researchers allow students to generate their own vocabulary and develop a concept map from the students’ chosen vocabulary. This method allows the researcher to ascertain what ideas the student is relating to a specific concept and how he/she is relating the ideas. Finally, some researchers create a concept map for the student based on data gathered about the student. A researcher would use concept maps in this way in order to help organize information about a particular student.

Recently, some researchers have investigated the validity of concept maps as a research tool in mathematics education (Latumo, 1994). Latumo found evidence to support her claim that student generated concept maps show validity as a research tool when compared to clinical interviews. Latumo (1994) compared the concept maps of 24 students to the students’ responses to interview questions. The interview questions were designed to “tap some of the ideas appearing on their concept maps” (pp. 63). The students’ concept maps and responses to interview questions were scored by Latumo and others, checked for inter-rater agreement, and then the score on each students’ concept map was compared to his/her interview score. The agreement in scores on the concept maps and interview questions ranged from 83.3% to 91.7%.
Concept maps were used in the current study as a means of validating my claims about students’ conceptualizations of rate of change, derivative, and integral. I asked each student to construct a concept map of rate of change, derivative, and integral, without providing the students a vocabulary list. Thus, it was my intention to gather information about what ideas the students related to a given concept and how the students organized their ideas.

At the end of the third interview, I introduced the students to concept maps. One student, Todd, had used concept maps previously in his schooling. I presented two examples of concept maps to the students, one dealing with the real-number system that was drawn by a team of pre-service teachers (Baroody & Bartels, 2000) and one dealing with the concept of function drawn by a college mathematics student (Williams, 1998). I answered any questions the students had about concept maps and then asked each student to construct a concept map for rate of change. I left the interview room while the students constructed their concept maps and allowed unlimited time for the construction of the concept maps. Each student constructed a concept map for derivative and integral during the final interview. Again, I left the room while the students constructed their concept maps and allowed unlimited time for the construction of the concept maps.

Data Analysis

In the tradition of Stake who claims that, "There is no particular moment when data analysis begins" (pp. 71), data analysis was an ongoing process throughout this study. The major source of data in this study was the student interviews. The audiotapes of the students’ interviews were completely transcribed and checked for accuracy.
throughout the study. Initial analyses of the interviews were used to inform future interview questions and help focus the classroom observations. Observation notes and notes taken during the clinical interviews were also transcribed.

Three main types of qualitative data analysis were employed in this study: the technique of microanalysis borrowed from grounded theory, within- and cross-case analyses, and triangulation of data. I will begin with a discussion of my microanalysis of the data.

Microanalysis

Micro-analytic techniques involve a detailed analysis and interpretation of the data. Strauss and Corbin state that, "Doing line-by-line [micro-analytic] coding is especially important in the beginning of a study because it enables the analyst to generate categories quickly and to develop those categories through further sampling along dimensions of a category’s general properties" (pp. 119). Since microanalysis takes a great deal of time and generates a large amount of data, micro-analytic techniques were used on only a portion of the data, specifically each student’s first interview. The transcripts of the students’ first interviews were chosen as the primary data source for the microanalysis since these were the earliest pieces of data collected. It was necessary to conduct a microanalysis on the earliest pieces of data since one of the research goals was to generate a classification scheme for the way students use physics to help them conceptualize calculus concepts. Conducting a microanalysis on the earliest pieces of data allowed me to develop such a classification scheme and further test and refine it with data collected later in the year.
Additionally, the interview transcripts provided data rich in detail, which is necessary for conducting a microanalysis. The transcripts were scanned for interesting and relevant paragraphs and I conducted a microanalysis of the selected paragraphs. Students’ words were considered both individually and as part of a more meaningful sentence. Multiple definitions and meanings were given to the words and sentences and my notes were re-read and checked for the introduction of researcher bias. As an example of the microanalysis, consider the following passage from Rob along with my analysis of Rob’s words. (Note: Rob’s responses are in bold and my analysis notes are in italics.)

And 1 and if it was equal to 1 that’d mean it’s continual rate. And it hasn’t changed at all.

Continual rate: He could mean ‘continuous’. It will always have a rate. It will always have a rate of 1. 1 is an important number here for Rob. A rate of 1 is a special rate. Rob’s next sentence perhaps gives a clue to what he means by ‘continual rate’: a rate that doesn’t change or a constant rate. So he could mean that the rate of change is not changing.

Change could mean: mutate; alter one’s form; differ; evolve; fix.

Cause if x is equal to 1 and y is equal to 1 there’s, the average rate would be 1, so it’s just, nothing changed.

Rob does not mention “change in x” or “change in y” here – he simply says that x = 1 and y = 1. He could mean change in x, but he does not mention the change in x. Rob said that “nothing changed”. Nothing became different? Nothing altered form? What’s nothing: no aspect of the graph has changed? No aspect of the function has changed? One times something will not change that something.

Micro-analytic techniques also were employed to analyze perplexing pieces of data as well as for generating provisional hypotheses. Perplexing pieces of data are data that appear to contradict provisional hypotheses or present a situation that does not fit into a provisional category. For instance, one student, Rob, gave an interesting interpretation of the Cartesian axes when presented with a graph of a function, f(x), and asked to find the
average rate of change between different values of x. Rob indicated that he was thinking about the x-axis as the ‘rate’ axis and the y-axis as the ‘distance’ axis. Rob proceeded to discuss his answers to the rate of change problems referencing the ‘rate’ and ‘distance’ axes. The segments of data that included Rob’s discussion of the ‘rate’ axis were singled out and microanalyzed in order to more clearly interpret Rob’s use of the ‘rate’ axis in this way. The use of micro-analytic techniques allowed me to consider the range of plausible interpretations of the data in question.

Physics Use Scheme

During the micro-analytic phase of analysis, it was my goal to generate provisional hypotheses concerning students’ uses of physics in their conceptualization of calculus concepts. The result of my microanalysis was the development of a scheme that classified each student according to his/her use of physics. The classifications are listed below with a short description of each:

**Non-users.** Non-users are those students who simply do not use physics, in any sense, to help them conceptualize calculus concepts. These students’ discussions of calculus problems involve non-physical vocabulary.

**Contextualizers.** Contextualizers are those students who not only discuss calculus problems in terms of physics, but also show evidence of immersing the problem in a physical context in order to solve it.

**Example-users.** Example-users are those students who refer to examples from physics to help them make sense of calculus concepts and problems. They do not contextualize the problem, that is, they will talk about physics in a way that is disconnected from the problem at hand. They also tend to use physical phenomena to make sense of an answer to a calculus problem.
Mis-users. Mis-users are those students who carry over misconceptions of physics concepts to misinform their conceptualizations of calculus concepts. Often misusers allow their misconceptions of physics concepts to dominate their thinking, even on abstract, mathematical problems.

The categories in the Physics Use Classification Scheme do not necessarily represent disjoint classes of physics use. For instance, the characteristics and features of a Contextualizer might overlap the characteristics and features of an Example-User. My intention was that the descriptions of Contextualizers, Example-Users, Non-Users, and Mis-Users would be modified and further developed as I continued to analyze the data. In fact, a new category emerged as three independent mathematics educators re-coded data in order to check for inter-rater agreement. The new category was labeled "Language-Mixers". Language-mixers are those students who tend to use language from physics in their discussion of calculus concepts. They use a concrete, physical language to discuss problems without contextualizing the problem or referring to an example. The Physics Use Classification Scheme will be discussed in more detail in a forthcoming section.

Summary of Microanalysis. The microanalysis of the data can be considered the first or initial stage of data analysis. During the initial, microanalytic stage, provisional categories emerged (for example, contextualizers and example-users) and these categories were subsequently tested in the next phase of analysis. The next phase of the data analysis was a within-case analysis of each student.

2 Later in the data analysis, the Misuser category was moved to the status of a sub-category. The researcher and others felt that none of the students in the present study exhibited Misuser tendencies on a regular basis. Rather, students' misconceptions would interfere with their mathematical conceptualizations as they were working on specific problems.
Within-Case Analyses

I had two main goals for the within-case analysis of each student: (1) Determine the proficiency of each student to work with specific types of representation (symbolic, numeric, physical, graphical. This part of the analysis will help determine if the results of the present research study are consistent with results previously reported in the literature. (2) Test the stability of the categories that emerged during the microanalysis. This part of the analysis investigates if the students consistently use physics in a certain way to help them understand calculus concepts. Tables 4 and 5 are copies of the coding schemes used to analyze the interview data.

<table>
<thead>
<tr>
<th>Representation</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Symbolic</td>
<td>Student uses formulas, mathematical expressions and symbols to solve the problem.</td>
</tr>
<tr>
<td>Numeric</td>
<td>Student uses data in a table or chart to solve the problem.</td>
</tr>
<tr>
<td>Graphical</td>
<td>Student uses graphs or pictures to solve the problem.</td>
</tr>
<tr>
<td>Physical</td>
<td>Student uses physical examples or physics concepts to solve the problem.</td>
</tr>
</tbody>
</table>

Table 4: Representation Coding Scheme

Table 4 is the coding scheme used to analyze the data according to the students’ use of representation. The descriptions of the representations were developed using definitions and examples from the work of Zandieh (2000) and various Calculus textbook discussions of function representations (Hughes-Hallet, Gleason, et. al., 1994; Ostebee & Zorn, 1992). This coding scheme was used to analyze transcript episodes, students’ homework assignments, students’ in-class activities, and students’ examinations.
<table>
<thead>
<tr>
<th>Physics Use</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Contextualizer</td>
<td>Student works and talks through problem as if it were a physics problem. Majority of technical vocabulary used to solve problem is physics terminology. Evidence that student is thinking about the problem in terms of physics.</td>
</tr>
<tr>
<td>Example-User</td>
<td>Student uses physics examples to justify solutions to problem or help make sense of a part of the problem. Actual problem at hand is solved using mathematical concepts. Student does not submerge the problem in a physics context. Majority of technical vocabulary is mathematical terminology.</td>
</tr>
<tr>
<td>Mis-User</td>
<td>Student’s use of physics misconceptions interferes with student’s solution of the problem. Student uses physics misconception to incorrectly solve problem.</td>
</tr>
<tr>
<td>Language-Mixer</td>
<td>Student intersperses physics and calculus terminology as he/she solves problem. Student does not immerse problem in physics context or use examples to justify solutions or help make sense of problem. Rather, student intermingles physics and mathematical language as he/she solves the problem.</td>
</tr>
<tr>
<td>Non-User</td>
<td>Student does not use physics concepts or language to solve problems.</td>
</tr>
</tbody>
</table>

Table 5: Physics Use Coding Scheme

The coding scheme presented in Table 5 was developed from the descriptions of the physics use categories so that I could code the transcripts more systematically. It also served as a reference for the independent raters who coded the transcripts. This coding scheme was used only to analyze the transcripts of the interview episodes since the students' physics uses were not immediately evident in other forms of data such as examination questions and homework problems. For example, if a student used the relationship between position, velocity, and acceleration to help him/her solve an antiderivative problem on an examination, the student’s written answer to the problem probably would not capture his/her thought process involving the physics concept and therefore would not be coded according to the student's use of physics.
The interview transcripts were first coded for the representations each student worked with. Note that all four interview transcripts were coded for each student, even the transcripts of the first interviews, which were used in the microanalysis. Although the interview tasks were presented to the students in one context the students often referred to other contexts as they worked through the tasks. For example, in the series of interview tasks, Derivative Task 1 – Derivative Task 4, I presented the students with graphs of functions without furnishing the explicit formula of the function. As some of the students worked through these tasks, they talked about the functions as formulas (symbolic) or as position curves (physical).

After the interview transcripts were coded for representation, I scanned and highlighted segments where the students talked about physical representations. Then I coded the data according to the classification scheme that evolved during the microanalysis. That is, if a segment was initially coded as a physics representation, that segment was more specifically coded for how the student was using physics: as a contextualizer, example-user, or mis-user. Then a copy of selected transcripts and student work, along with the interview instruments, were given to three independent raters who used the criteria to re-code the data. The results from the rater’s coding were then compared to my original coding. I talk about the results of the inter-rater coding in the next section.

Inter-Rater Reliability

Three independent mathematics education researchers were given copies of selected transcripts, student work, and interview tasks and re-coded the data according to the researcher’s coding schemes. The mathematics education researchers’ codes were
compared to my original codes. A percentage of coding agreement was established by
counting the number of episodes that the independent raters' codes matched my codes
and dividing that by the total number of episodes coded. For the first layer of coding —
coding for student representation use — the independent raters agreed with my original
coding 89% of the time. On the second layer of coding, coding for physics use, the
independent raters agreed with the my original coding 91% of the time.

As the independent researchers coded for physics use, a new category of physics
use emerged: Language — mixers. Language-mixers are those students who tend to use
language from physics in their discussion of calculus concepts. They use a concrete,
physical language to discuss problems without contextualizing the problem or referring to
an example.

Cross-Case Analysis

The third phase of the data analysis was a cross-case analysis. The purpose of the
cross-case analysis was to uncover themes common to all cases. Specifically, I looked at
comparing students' performances on interview tasks and selected homework,
examination, and in-class activity problems. My goal in performing the cross-case
analysis was twofold: (1) Identify characteristics of the group of eight students, as a
whole, and compare those characteristics to descriptions of students previously discussed
in the literature. This part of the analysis allows me to make statements about the
conceptualizations of the eight students in the present study relative to previously
reported results. (2) Generate hypotheses about the characteristics of those students who
were classified as Contextualizers, Example-Users, Language-Mixers, and Non-Users.
This part of the analysis helps investigate the manner in which students use physics to aid
in their conceptualization of calculus concepts. I discuss how each student was classified according to physics use in the next section. The results of the cross-case analysis are discussed in Chapter VI.

Physics Use Classification Scheme

The Physics Use Classification Scheme was developed through the microanalysis of the eight students' transcripts from the first interview and refined during the Within-Case analysis and Inter-Rater coding. The Physics Use Classification Scheme was used in two ways during the data analysis. First, it was used as a scheme to code episodes of the students' interview transcripts. The coding scheme is presented in Table 5 above. Episodes of the interview transcripts were examined for evidence of the students' use of physics based on the scheme in Table 5 and coded accordingly.

Next, the Physics Use classification scheme was used to categorize each student according to the manner in which he/she used physics to solve calculus problems. I scanned the interview transcripts for the physics use codes. Then I counted the number of times each code appeared. I classified each student based on the largest number of instances of a particular code. For example, Rob's second interview had 12 episodes coded as Physics-Contextualizer, two episodes coded as Physics-Example-User, one episode coded as Physics-Misuser, and one episode coded as Physics-Language-Mixer. I tallied the results from Rob's second interview with his other three interviews and found that the majority of physics episodes were coded Contextualizer. If a particular student had relatively few (1-2) episodes in the final tally coded as using physics, then that student was labeled a Non-User.
I attempted to classify each student as either a Contextualizer, Example-User, Non-User, or Language-Mixer for all problems as described above. As I analyzed the data, I realized that the manner in which many of the students used physics to solve average rate of change problems differed from the way that they used physics to solve derivative and integral problems. Thus, I classified each student according how he/she used physics to solve average rate of change problems and then re-classified each student according to how he/she used physics to solve derivative and integral problems. Then, I examined the students’ in-class activities, homework, and examinations to find evidence to corroborate the initial classifications. The classifications of each student, along with the supporting evidence are presented in Chapter V.

Validity

I am treating validity as an issue distinct from data analysis much in the spirit of Joseph Maxwell (1996). Maxwell claims that validity is the final component of any qualitative research design, separate from methods since validity threats are made implausible by evidence, not methods. Maxwell identifies three major threats to the validity of a qualitative research study. I have outlined Maxwell’s three threats below and how the present study deals with the threats:

1. Threats to valid description. Maxwell claims that the main threat to valid description is inaccurate or incomplete data. Threats to valid description can be avoided by audio or videotaping interviews and observations and transcribing these recordings. As described in the Data Analysis section above, each student interview was transcribed
and the transcriptions checked for accuracy. I took detailed notes while observing the class and immediately retyped the notes after each classroom observation.

2. Threats to valid interpretation. Maxwell claims that the main threat to valid interpretation is not allowing the data to speak for itself, by imposing one’s own meaning on the data. Threats to interpretation can be avoided by systematically checking how subjects in the study make sense of what’s going on and being aware of one’s own assumptions and biases. During the microanalytic phase of the data analysis, I analyzed my own analysis of the data to check for the introduction of researcher bias or interpretation into the analysis stage. I attempted to assign multiple definitions to the students’ words, as the microanalysis technique dictates, thus considering alternative explanations.

3. Threats to theory. Maxwell claims that the main threat to theoretical validity is not considering alternative explanations or ignoring discrepant data. As stated in the second point, alternative explanations were considered through the use of microanalytic techniques.

Wolcott (1994) also offers a number of suggestions for researchers to satisfy the challenges of validity as outlined by Maxwell above. These suggestions include talking little and listening a lot, accurately recording and reporting data, and fully reporting data. During the student interviews and in-class observations, I gave very little feedback to the students and encouraged them to talk through their solutions to problems as much as possible. The interview transcripts were checked for accuracy and portions of the in-class observation notes were checked against the students’ class notes. Additionally,
drawing on a number of data collection sources allowed me to report the data as completely as possible.

Finally, Maxwell (1996) discusses the strategy of triangulation for ruling out validity threats and increasing the credibility of one’s conclusions. Triangulation is a process of collecting data from a wide range of sources, using a variety of methods. Maxwell claims that, “This strategy reduces the risk of chance associations and of systematic biases due to a specific method and allows a better assessment of the generality of the explanations that you develop” (pp. 94). Data was collected from multiple sources (interviews, classroom observations, and student work) in order to effectively substantiate conclusions and address issues of validity.

**Generalizability**

“The generalizability of qualitative studies usually is based, not on explicit sampling of some defined population to which the results can be extended, but on the development of a theory that can be extended to other cases” (Maxwell, pp. 97). Maxwell’s statement serves to quell the criticism that qualitative research studies are never generalizable beyond the specific setting or persons studied. Certainly, one does not expect to make the type of generalizations and extrapolations that probabilistic sampling allows. However, as Maxwell claims in the above statement, theory, rather than results can be generalized from qualitative studies. That is, even though the results of a qualitative study may not be generalizable to other populations and settings, the theory underlying the results can be employed in a range of situations. In order to
understand this distinction, it may be useful to consider the different types of
generalizability.

Stake (1995) proposes the existence of two types of generalizability. One type of
generalizability is propositional, the scientific definition of generalization, and the other
is more intuitive and empirical, based on experience. Stake (1995) calls the latter type of
generalization naturalistic generalizations. “Naturalistic generalizations are conclusions
arrived at through personal engagement in life’s affairs or by vicarious experience so well
constructed that the person feels as if it happened to themselves” (Stake, 1995, pp. 85).
Theory developed in a qualitative study can be extended to other situations through a
process of naturalistic generalization.

Stake (1995) discusses a number steps for consideration to assist the reader in
making naturalistic generalizations. I have outlined some of Stake’s steps below and how
the present study addresses these steps:

1. Provide adequate raw data prior to interpretation. Providing raw data will allow the
reader to consider his/her own interpretations. Chunks of raw data are provided in the
presentation of the eight cases in Chapter V. The chunks of raw data are taken from
the student interviews, homework assignments, in-class activities, examinations, and
my observation notes.

2. Describe the methods of data collection and analysis used in ordinary language. This
description should include a discussion of triangulation. The methods of data
collection and analysis were presented and discussed in this chapter. The
presentation of the data collection and analysis for the present study included a
discussion of triangulation.
3. Make available information about the researcher and other sources of input.

Information about my involvement with the Calculus/Physics program as a teaching assistant, evaluator, and classroom participant-observer was discussed in the present chapter.

Summary

The methods of data collection and analysis were discussed in this chapter. Data was collected through semi-structured, task-based interviews, participant-observation in the Calculus/Physics classes, and students' examinations, homework assignments, in-class activities, and class notes. The data analysis consisted of three phases: microanalysis, within and cross-case analysis, and triangulation.

The microanalysis of the data can be considered the first or initial stage of data analysis. Conducting a microscopic analysis on selected pieces of data allowed me to let the data "speak" rather than forcing the data into fitting my own theoretical conclusions (Strauss & Corbin, 1998, pp. 65). During the initial, microanalytic stage, hypotheses of students' uses of physics emerged and these hypotheses were subsequently tested in the second phase of analysis.

The second phase of the data analysis was a within-case analysis of each student. The interview transcripts were first coded for the representations (numeric, symbolic, graphical, and physical) each student used to solve problems. Then the transcripts were re-coded using the physics use classifications. The purpose of the re-coding was to determine how the students were using physics to
inform their conceptualizations of calculus concepts. I also conducted a cross-case analysis of the data. The purpose of the cross-case analysis was to uncover themes common to all cases.

Copies of selected transcripts and student work, along with the interview instruments, were given to three independent raters who used the developed criteria to re-code the data. The results from the rater's coding were then compared to the original coding.

Finally, students' examinations, homework assignments, and in-class activities were used to corroborate conclusions drawn from the interview data. The results of the analysis are presented in subsequent chapters.
CHAPTER V

ANALYSES AND RESULTS

Introduction

This chapter presents and discusses the results of applying a physics use classification scheme to the data collected for each of the eight students. Recall that the physics use classification scheme resulted from the analysis of the data and addresses the main research question: How do students draw upon physics concepts to inform their understanding of calculus concepts? I will present an overview of the physics use classifications assigned to each student, followed by a presentation of the eight cases.

The presentation of the cases includes evidence supporting the physics use classifications and also discusses and summarizes the student participants' conceptualizations of average rate of change, derivative, and integral and how they relate to the students' physics use classifications. These conceptualizations were gathered from students' remarks and actions as they responded to interview tasks, students' responses to problems presented in homework assignments, on in-class activities, and on examinations, and my observation of students during the class. In this section, each student is treated as a separate case and within each case the data is organized as indicated in the following outline.
I. Background

This section gives general information about the student as gathered from the students' responses to the Background Questionnaire (see Appendix C) and from my informal discussions with the students. Included is a description of the students' experiences in high school calculus and physics classes and the students' grades in the fall semester of the Calculus/Physics class.

II. Physics Use Classifications

This section presents the results of the physics use classification for the student as well as supporting evidence for that classification within the contexts of average rate of change and derivative and integral. Additionally, this section includes a discussion of students' misconceptions of physics concepts and the effect of those misconceptions on the students' conceptualizations of average rate of change, derivative, and integral.

III. Concept Image

This section presents my interpretation of the students' concept images for average rate of change, derivative, and integral as well as evidence supporting my interpretations. Included is a description of the students' ability to work with the concepts of derivative and integral in graphic, physical, symbolic, and numeric mediums. This section also includes a discussion of the students' conceptualization of the Fundamental Theorem of Calculus.

As described in previous chapters, problems were presented to the students in various mathematical contexts during the interview sessions and on homework assignments, in-class activities, and examinations. These mathematical contexts include
physical, graphic, symbolic, and numeric mediums. Recall that the definition of representation used in the present study is representation as a mathematical context that is used by a student to express a conception. Although the problems in the interview sessions and on homework assignments, examinations, and in-class activities are presented to the student in a specific context or medium, the student may have chosen to solve the problem working in a different representation. For example, a student encounters a problem such as the one pictured below in Figure 6.

*Sketch a graph of the derivative of the following function:*

![Graph](image)

*Figure 6: Sample Graph Problem*

Although the problem asked the student to sketch a graph of a derivative (graphic context), the student chose to represent the problem symbolically and solved the problem by first finding a formula for the function and then taking the derivative using derivative algorithms. In this case, the problem was presented graphically, but the student represented the problem symbolically. This is evidence of the student's strong symbolic presentation of her concept image of derivative.
While I make a distinction between the physical, graphic, symbolic, and numeric contexts, often problems were presented to the students using more than one context at a time. For example, the students often encountered problems prompting them to sketch position and acceleration graphs given a velocity graph. Because the problem asks the students to sketch a graph, the context of the problem is graphic. But the problem also involves the physics concepts of position, velocity, and acceleration, thus adding a physical dimension to the problem. The context that the problems were in was not of interest in the present study. Rather, the representation that the student used to solve the problems was the focus of the data analysis.

Physics Use Classification Scheme

The physics use classification scheme was developed through the microanalysis of the eight students' first interview transcripts and refined during the within-case analysis and inter-rater coding of the data. The physics use classification scheme addresses the main research question: How do students draw upon physics concepts to inform their understanding of calculus concepts?

As discussed in the previous chapter, each student's work on average rate of change tasks was analyzed separately from his/her work on derivative and integral tasks. Each student received two physics use classifications: one to describe how the student draws upon physics to inform his/her understanding of average rate of change and one to describe how the student draws upon physics to inform his/her understanding of derivative and integral. The physics use classifications for each of the eight subjects in the present study are presented in Table 6.
Table 6: Student Physics Use Classifications

The presentation of the cases in the current chapter provides supporting evidence for each student’s physics use classifications. The cases also present a discussion of the students’ ability to work with average rate of change, derivative, and integral in multiple contexts. The focus on each student’s ability to work with the calculus concepts in varying contexts helps address the research question: Do students in the present study possess conceptualizations of calculus concepts similar to those previously documented in the literature? Furthermore, the analysis addresses one of the goals of the present study which is to examine students’ concept images of average rate of change, derivative, and integral. What follows is a presentation of the cases, beginning with Rob.

Rob

Background

Rob is an Electrical Engineering major who decided to major in Electrical Engineering because of his interest in electronics and fixing things. Rob reported that he also enjoys mathematics and computer science and that his interest in those subjects was also a factor in choosing his major. In March, 2001, Rob reported that he was happy with
his major so far and that he did not plan on changing his major from Electrical Engineering.

Rob’s hobbies include bicycle riding, reading and computer gaming. Rob is also involved in the Fencing Club on campus. During his first semester, Rob was enrolled in *Perspectives in Electrical and Computer Engineering, Classical Mythology,* and *Calculus/Physics.* In his second semester, Rob was taking the *Calculus/Physics* class as well as *Introductory English* and *Introduction to Scientific Programming.* Rob reported that he is looking forward to taking more major-oriented classes during the next few years. Rob seems to take his education very seriously. He stated, “My goal is to do the best I can with my classes. I know that good grades are a very high determining factor for the quality of the job that hires the student.”

Rob reported that he decided to enroll in the Calculus/Physics program because the idea of a small class appealed to him. He believed that due to the small student-teacher ratio, the professors would get to know each student individually and he viewed this as an incentive to enroll. Rob also stated that he believed the pace of the curriculum moved “much faster” than the pace of the separate, lecture sections of Calculus and Physics and this also was an attractive feature of the Calculus/Physics program.

During the summers, Rob works at a hospital as a CAD operator. Rob’s major duties involve converting paper blueprints into computerized blueprints. Rob reported that he enjoys this job especially because he sets his own hours and works at his own pace. Rob did not have an outside job during the 2000-2001 academic year.

Rob reported that he had taken calculus during the first semester of his senior year in high school. His calculus class was a block-scheduled class enrolling approximately
15 students. Rob reported that the class covered topics involving limits, derivatives, and integrals during the semester. The calculus class was a mixture of lecturing and group work. Rob's physics class was scheduled immediately after his calculus class during the first semester of his senior year in high school. During the semester, the class covered topics such as acceleration, force, resonance, and electric circuits. Rob reported that he didn't like working with forces, but enjoyed learning about sound and liked the lab portion of the physics class. Rob specifically mentioned enjoying a lab dealing with friction in which the class used sleds to examine properties of friction.

During my first interview with Rob, I noticed that Rob seemed to depend on his memory of formulas to answer questions. On a number of occasions, he mentioned that, "some people spent a lot of time deriving these." Rob was referring to the fact that some famous mathematicians had spent a great deal of their lives deriving and refining the mathematical formulas that we take for granted today.

Rob received a B- in the first semester of calculus. Twenty five out of 48 students in the Calculus/Physics class received a grade in the range of B- to B+. Rob also received a B- in his first semester of physics.

Physics Use Classification

Overview. Rob was classified as a Contextualizer in the categories of Average Rate of Change and Derivative and Integral. Rob was classified as a Contextualizer in both categories because his internal images of average rate of change, derivative, and integral were frequently manifested in physical representations. Rob often used physics concepts to describe his presentations of average rate of change, derivative, and integral. As he worked on problems during the interview sessions, Rob primarily spoke about the
problems in a physical representation. That is, Rob worked through the problems as if they were physics problems, evoking physics concepts and using his knowledge of physics to solve the problems.

Unlike most of the other eight students, Rob used physics consistently as he solved average rate of change and derivative and integral problems. That is, Rob was classified as a Contextualizer in both the categories of Average Rate of Change and Derivative and Integral. Rob was one of the weakest of the eight students participating in the present study. I often observed Rob working at a slower pace than other students in the Calculus class. Rob frequently chose to work on the calculus in-class activities by himself, rather than collaborate with his partner. However, I observed Rob to be more interactive with his group members in Physics class. Rob frequently took part in group and whole-class discussions. I believe that Rob felt more comfortable with the physics concepts than the calculus concepts discussed during the first semester of the Calculus/Physics class since he frequently talked about physics concepts as he solved calculus problems. Furthermore, I observed him as a more active participant in the physics class than in the calculus class. Rob's comfort working with and discussing physics concepts may be a reason for his tendency to use physical representations when solving calculus problems.

The next two sections present evidence for Rob's classifications as a Contextualizer in the categories of Average Rate of Change and Derivative and Integral. Many of the examples show that Rob solved average rate of change, derivative, and integral problems using the physical representation.
Average Rate of Change: Contextualizer. On many occasions, Rob immersed average rate of change problems in a physics context. That is, Rob talked through his solutions to average rate of change problems as if they were given to him in a physics context. For example, consider how Rob began talking through his solution to Average Rate of Change Problem 5:

Well this I wasn't entirely sure, so I just, the first thing I did, I just graphed um, one of these must be rate... one of these axes is rate, the other one is distance. So this one is probably distance, the y axis. The horizontal [axis] would be the rate.

Notice that Rob talked about the vertical axis as representing distance and horizontal axis as representing rate. Rob’s designation of the horizontal axis as rate created problems for him as he worked through Average Rate of Change Problems 5 through 8. As Rob continued to solve Average Rate of Change Problem 5, he again referred to the vertical axis as distance.

They use the distance, so it would be 4 and 1. And what you’d basically do is, I think, just like a slope... it would be the change in the distance, so four minus one would give you three. And put that over the change in the rate, I think. The change in x. Which would be 2 and 1, so 2 minus 1 that would give you 1. So it’s 3 over 1. So it’s about three times, I guess. Something like that.

Notice that Rob stated, “They use the distance...”. Rob’s use of the word ‘they’ indicates his thinking that the problem was stated in terms of distance, in a physical context. Rob read the problem as if it were presenting data in terms of the distance an object or person had traveled.

The next passage is an excerpt of Rob’s work on Average Rate of Change Problem 10. As he began to solve the problem, which was presented as the position...
versus time graph of a car, Rob used the physical context to make sense of the shape of the graph.

R: Now this is telling me is that the object is accelerating and goes a certain distance, and then goes in reverse to a negative point and then it starts to go back forward again. So if a car is driving then it would back up for a while and it’s going forward again.
I: OK. And how do you know that? How did you know that just from looking at the graph?
R: Well the way the graph is, it has positive values and negative values and negative values is when it’s going either to the left or backwards from a starting point.

Rob used the vocabulary ‘accelerating’, ‘goes in reverse’, and ‘go…forward again’.

These phrases are indicative of Rob’s use of his past experiences to interpret the motion of the graph. Rob seemed to be using the word ‘accelerating’ here to mean that the velocity is greater than zero. Trowbridge and McDermott (1981) found that students often confuse the concepts of velocity and acceleration. Trowbridge and McDermott (1981) claim that students’ experiences with velocity and acceleration in real life may contribute to their confusion of the two concepts. At other points in the interviews and in his class work and homework, Rob seemed to exhibit the velocity-acceleration confusion that Trowbridge and McDermott identified in 1981. More examples of Rob’s velocity-acceleration confusion as well as other physics misconceptions will be discussed in a later section.

Rob’s answers to the Average Rate of Change interview tasks, along with his in-class work and examinations indicate that Rob’s presentation of average rate of change was largely physical. Rob used the physical representation to solve many average rate of change problems. Early in the semester, Rob exhibited a tendency to confuse the concepts of velocity and acceleration, a difficulty previously documented in the literature.

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(Trowbridge & McDermott, 1981). A more extensive discussion of Rob’s conceptualization of average rate of change is presented in a forthcoming section.

**Derivative and Integral: Contextualizer.** Rob tended to contextualize derivative and integral problems as he worked through them. Rob’s conceptualization of the derivative and integral as something used in both calculus and physics is evident in his concept maps. (See Appendix E for a copy of Rob’s Derivative and Integral concept map.) On his Derivative concept map, one branch of the term ‘derivative’ is a calculus branch and one branch is a ‘physics’ branch. In the calculus branch, Rob used position, velocity, and acceleration as examples of function-derivative relationships. Rob also used graphs of position, velocity, and acceleration to talk about derivatives on the physics branch. Furthermore, Rob talked primarily about kinematics when I asked him to describe the relationship between a function and its derivative:

Well a function, I just see it as the physics part, I guess. Um, the position is when you take the derivative of it you get the velocity, when you take the derivative of that you get the acceleration. And also if you have acceleration you take the anti-derivative you get your velocity and so on.

Rob had a strong tendency to contextualize graphical problems in terms of physics. On the two graphical derivative tasks that I gave Rob during the second interview (Derivative Tasks 1 and 3), he began talking about the problems in terms of kinematics. For example, as he began to solve Derivative Task 1, Rob said:

And you’d assume that this is, in terms of physics, this would be position. And you’d be trying to solve what’s happening with the velocity. So, position it seems to be increasing and it’s falling well slowly...like in terms of t, it’s increasing it’s, it seems to be slowing down and then it turns around and goes backwards and then it goes down and comes back up again. So I want to show that in my graph.

Rob even talked about a specific physical situation in his solution to Derivative Task 1:
But to me it seems that...if it was a ball that you pushed across a table and it was, a distinct v...I’ll label these t and these v...distance... so times time, distance is decreasing and then it stops, turns around, goes backwards, time is still going...so in terms of velocity, that could be positive.

Further evidence of Rob’s use of physics to help him interpret graphs occurred when I presented him with Derivative Task 3. Rob faltered as he tried to sketch a graph of the function, given a graph of its derivative. He attempted to start graphing the function using zeros and points of inflection on the derivative graph, but struggled to make sense of the graph. When I prompted him to consider the graph as the velocity of a car, he immediately talked through a solution:

I: So what if I told you that this is a graph of velocity of a car and I want you to produce the position graph. Does that help at all?
R: Um, if there was a velocity, it would be slowing down, so it’s decreasing. Um... I can reflect that by showing...this starting somewhere...(pause)...It would, position would still be increasing and then it goes negative once it gets here because, it’s a negative velocity for that time.
I: OK.
R: So you’d probably want to go down and then go up.

Notice that Rob connects the concept of decreasing velocity with the experience of ‘slowing down’. Rob also alludes to the prominence of real-life experiences in his conception of derivative in his concept map of Derivative (Appendix E). In his concept map, Rob gives an example showing the relationship between position, velocity, and acceleration graphs and links the graphs with the statement, ‘Used for real life problems’. It seems that Rob connects the physical concepts of position, velocity, and acceleration with real-world experiences.
However, contextualizing calculus problems did not always benefit Rob. In Derivative Task 5.1, Rob faltered as he tried to interpret and answer the question. He decided to draw in the graph of the derivative and figure out the answer from his graph.

R: If this is just position – I’d have to actually draw in the derivative and then figure it out from there? (pause)
I: What do you think?
R: That’s probably what I have to do. (pause, mumbling, drawing graph of derivative). And it seems like that would start – ‘cause it starts out as, ah, it’s starting in the negative direction. So we’d have to start with a negative velocity or negative or a negative grade. Then it would go up, past, passes zero… increasing with a (garbled). And then it starts getting faster. It turns at 1 and goes back up to the zero...

Then Rob looked at the rate of change of the derivative graph to answer the question and decided that the answer must lie between points E and H, based on his derivative graph.

In addition to inappropriately contextualizing calculus problems, like in the above example, Rob also possessed certain physics misconceptions that influenced his thinking about other calculus concepts. Consider Rob’s answers to the following examination problem, shown in Figure 7. Rob correctly sketched a position versus time graph and a velocity versus time graph of the motion. However, Rob’s acceleration versus time graph and accompanying explanation of his acceleration versus time graph uncover Rob’s misconception that if an object is speeding up than the object’s acceleration is greater than zero. Notice that Rob correctly matched up the maximum on the velocity graph with a zero on the acceleration graph. However, Rob’s misconception that speeding up is equivalent to positive acceleration caused him to sketch part of his acceleration versus time graph incorrectly.

Furthermore, during a group discussion in physics class, Rob’s group came to the consensus that the acceleration of a ball rolling on an inclined plane must be negative
because the velocity of the ball was negative. The group members continued to work under this assumption until the physics instructor prompted them to think more carefully about the relationship between acceleration and velocity. The group eventually decided that only when acceleration acts against velocity can an object slow down and ultimately come to rest.

Finally, Rob contextualized many of the integral problems. When I ask Rob to interpret his answer to Integral Task 2 (compute \( \int 3x \, dx \)), he once again evoked physics concepts:

I think of this probably as.... well just something you use for doing the physics part of calculus. 'Cause when you take the anti-derivative... 'cause you know that, like position, um you take the derivative of this equation, you take the derivative of it and you get the velocity and the derivative of velocity is the acceleration. And it works the same way if you go backwards. The anti-derivative of the acceleration is the velocity. Anti-derivative of this is...position.

Rob appeared most comfortable solving derivative and integral problems as if they were physics problems.

**Summary.** Rob solved many average rate of change, derivative, and integral problems by invoking a physical representation. Rob’s extensive use of the physical representation indicates that his concept image was largely made up of physical presentations of conceptions of average rate of change, derivative, and integral. Rob also possessed some physics misconceptions that sometimes hindered his ability to successfully solve calculus problems. In the next section, I discuss my interpretation of Rob’s concept images of average rate of change, derivative, and integral.
A student walks beside a 2-meter measuring stick, beginning her walk at the origin. Then she moves with decreasing speed toward the 2 meter mark. After coming momentarily to rest near the 2 meter mark, the student immediately begins moving toward the 0 meter mark with increasing speed. For each of the plots below, sketch graphs of this motion and briefly explain why you drew the plots as you did.

Brief Explanation:

The student walks at a given velocity then slows down to a stop. The student then turns around and moves in a negative direction.

The student is moving at a velocity, then the student slows down and moves in the negative direction carrying a velocity to be negative.

The acceleration is positive then negative as the student slows down. The student speeds up when she turns around and moves in the other direction.

Figure 7: Rob's Examination Problem Demonstrating Physics Misconceptions

Concept Image

Overview. In this section, I will discuss Rob's concept images of average rate of change, derivative, and integral. I attempted to re-construct Rob's concept images by
using his concept maps as well as his responses to interview tasks, homework assignments, examination questions, and in-class activities.

**Average Rate of Change.** Rob’s concept image of average rate of change was dominated by his physical presentation of average rate of change. Rob’s concept image of average rate of change focused largely on average rate of change as a numerical average. Furthermore, Rob did not appear, at least initially in the semester, to make a connection between average rate of change and slope of the secant line. The symbolic and numeric mathematical contexts do not become representations for Rob. Rather, he represented most problems as physical.

Rob initially appeared to have a disconnected conception of the average rate of change between two points and the slope of the secant line between two points. During the first interview, Rob attempted to solve most average rate of change problems by averaging values and using a notion of betweenness. For example, Rob solved Average Rate of Change Problem 10 by using the formula Distance = Velocity * Time to find the velocity of the car between time intervals of .5 seconds and then averaged the velocities of the half-second intervals to find the average velocity of the car between one and three seconds. After Rob solved Average Rate of Change Problem 10 in this way, I prompted him to draw the secant line between one and three. The next passage is the conversation that ensued:

I: Can you draw the secant line between t = 1 and t = 3?  
R: t = 1, so it would be up here... t = 3... So I’m just connecting the dots.  
The definition of a secant line is it touches only two points on the curve.  
I: OK. Great. Can you calculate the slope of the line?  
R: Um-hum. Yeah, this is probably the shortcut! The slope of the secant line is the average rate of change.  
I: So how do you know that?  
R: Um, someone told me and it just stuck in my head, I guess.
Notice that Rob called the slope of the secant line a ‘shortcut’ to finding the average rate of change between two points. Furthermore, Rob justified the claim that the slope of the secant line gives the average rate of change by stating, “someone told me”.

During the first interview, Rob did not recall that the slope of the secant line was the average rate of change until I prompted him to consider the secant line as he worked on Average Rate of Change Problem 10. However, Rob correctly computed the average rate of change on a homework assignment that he completed prior to the first interview. Rob was asked a similar question to the one I presented him with, except that the axes of the graph were labeled position and time. See Figure 8 for a copy of Rob’s answer to the homework problem.

There is evidence from his answer to this homework problem that Rob used a slope formula to compute the average velocity between two different times. First, Rob wrote: 
\[ m = \frac{\Delta P(t)}{\Delta t}; \quad m = \frac{P(t_2) - P(t_1)}{t_2 - t_1} \] as he estimated the average velocity of the object. Rob was clearly using slope (m) to calculate the average velocity in this problem. Rob also drew in a secant line on the graph connecting the 1 second and 3 seconds points. Furthermore Rob was able to use his answer to the average velocity problem to find the equation of the secant line between the same two time points. Although Rob appropriately used the slope of the secant line to compute average rate of change in this homework problem, Rob’s responses to interview tasks indicated that he did not have a strong presentation of average rate of change as the slope of the secant line.

Rob did not always make a connection between average rate of change and slope of the secant line as evidenced during his first interview with me. However, it seems that by the end of the semester, Rob made some connections between the slope of the secant
13. The position \( p(t) \) of an object moving along a line is given by the graph below. [2, p. 25]

(a) Estimate the average velocity of the object between times \( t = 1 \) and \( t = 3 \).

\[
\frac{\Delta p}{\Delta t} = \frac{p(3) - p(1)}{3 - 1} = \frac{2 - 1}{2} = \frac{1}{2}
\]

(b) Find the equation of the secant line of \( p(t) \) between times \( t = 1 \) and \( t = 3 \), and sketch the graph of the secant line on the plot above.

\[
r(t) = \frac{1}{2} t + \frac{1}{2}
\]

(c) Write down the formula you used to find the slope of the secant line in part 13b. Compare the formula with the one you used to find the average velocity in part 13a.

\[
\frac{\Delta p}{\Delta t} = \frac{p(3) - p(1)}{3 - 1}
\]

Figure 8: Rob's Use of the Slope Formula in a Homework Problem

Figure 8: Rob’s Use of the Slope Formula in a Homework Problem

line and average rate of change. Late in the first semester when Rob drew a concept map for Rate of Change, he indicated that the average rate of change is defined as

\[
\frac{f(x + h) - f(x)}{h},
\]

and pointed out that it was the slope of the secant line between two points on a curve. (See Appendix D for a copy of Rob’s Rate of Change concept map.)
Rob mentions in two other places on his concept map that the slope of the secant line is the average rate of change. Rob’s initial disconnect between average rate of change and slope could be due to his strong conception of average rate of change as a numerical average.

In the beginning of the fall semester, Rob solved many average rate of change problems by finding the numerical average of rates that he calculated over a number of intervals. Consider Rob’s discussion of his answer to Average Rate of Change Problem 5 below. Note that Rob named the y-axis ‘distance’ and the x-axis ‘rate’ to solve Average Rate of Change Problems 5 through 8.

So I look at the graph when x is equal to one, which would be the horizontal, so I look at one, and it's about here. I draw a little dot. And when x is two I go, when x is equal to 2 and then I look at the corresponding distance, so that's at 4. So it's between 4 and 1. So I just kind of took a halfway point and I used that as ah, the, average. That's at about 1.5.

In this passage, Rob equated halfway with average. The average is the halfway mark or point. Rob’s answer of 1.5 made sense to him since he named the x-axis ‘rate’. Thus, the average rate is the point halfway between 1 and 2. Rob averaged 1 and 2 to get 1.5. It seemed to me that Rob was ignoring the y-axis as he computed the average rate of change. In the next passage, I prompted Rob to further explain how he is using the word ‘average’ in this context. He used the formula for computing slope to answer my question here:

I: OK, so when you pick the halfway point, um, you said you pick the halfway point and it's like the average, can you explain to me a little bit about what you mean by that? So, the average of which numbers?
R: They use the distance, so it would be 4 and 1. And what you'd basically do is, I think, just like a slope...it would be the change in the distance, so four minus one would give you three. And put that over the change in the rate, I think. The change in x. Which would be 2 and 1, so 2
minus 1 that would give you 1. So it's 3 over 1. So it's about three times, I guess. Something like that.

I further prompted Rob to explain what “three times” meant. Rob started to second guess his use of the slope formula and it seemed that a previous notion of average as a middle value started to surface at this point.

I: OK. Um, can you, so, you were saying something like "it was three times". Can you explain a little bit about what you mean by that? So what does this answer mean to you?
R: (pause) OK. Well, it's 3 over 1...it would equal 3. And 1 and if it was equal to 1 that'd mean it's continual rate. And it hasn't changed at all. Cause if x is equal to 1 and x is equal to 1 there's, the average rate would be 1, so it's just, nothing changed. Since the rate changed from 1 to 2, it's, the answer's greater than 1 but less than 2. (pause). That wouldn't work (scratching out previous answer of 3). (long pause). It's either between.... I don't know how to calculate this. I'm not really sure right now.

Here, Rob indicated a notion of average that seemed to mean 'between'. He rejected his previous answer of 3 because it did not lie between 1 and 2. He stated that the 'rate changed from 1 to 2.' It would seem here that from Rob’s perspective, the y-values have no bearing on the answer. The rate changed along the x-axis. Rob ends up abandoning this idea; he was not sure how to proceed. But, he didn’t seem to realize that part of the problem might have been his labeling of the x-axis as 'rate'.

Rob proceeded through Problem 6 as he did for Problem 5. Looking at Figure 9, Rob darkened points in on the graph at (-1, 1) and (2, 4). When the secant line is drawn between those two sets of points, (0, 2) seems to be a reasonable midpoint.

I: OK. So you’re saying that, so you’re answer “twice”...so you picked a number that was in between negative 1 and 2.
R: Um-hum.
I: So what would that be? You can show me on the graph.
R: Uhh...plot -1, where it is on the graph...2...up here. So somewhere on the graph, about halfway, I guess, between these. Somewhere around there. Maybe 2? So that's probably where I got the “twice”.

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Questions 5 thru 8 refer to the following graph:

![Graph of a curve showing points at various x and y values.]

Figure 9: Rob's Response to Average Rate of Change Problems 5 - 8

Here it seems as if Rob was looking at the y-values. He seemed to be saying that 2 is in between the y-coordinate of -1 and the y-coordinate of 2. He didn't seem to notice that he was confusing notions of rate as along the x-axis (his definition) and rate involving a y-coordinate (closer to the standard definition, but not quite there, either).

On the next question, Rob broke from his previous pattern and referred to the slope of the secant line between (-3, 9) and (3, 9):

I think that would be none. 'Cause it starts at negative 3 and it goes up to 9. And on 3, on the positive side for x, it's the same number, so there would be no change at all. So you would just draw a line in between the points.

Here Rob seemed to abandon his idea about a value in the middle of an initial and final rate value. He could have meant here that 0 is the average of -3 and 3, but he gave no indication of thinking in terms of an average, as he had previously. He talked here about drawing a line in between the points – evoking a graphical image.
Rob also exhibited his tendency to solve average rates of change problems by computing an average on homework problems. For instance, Figure 10 shows Rob's work on a homework problem asking him to compute average velocities for a car over various time intervals. Notice that Rob wrote down the formula $d = vt$, but he solved the problem by adding the velocities and dividing by the total number of time intervals. Rob seemed to have a conceptualization of the term 'average' as an indication to add and divide by the total that dominates his thinking about average rate of change.

Although in the beginning of the semester, Rob approached average rates of change problems by computing a numerical average, by the end of the semester, he regularly used the formula $\frac{f(b) - f(a)}{b - a}$ to solve average rate of change problems. On the final examination, Rob correctly computed the average rate of change of $f(t)$ from $t = 1$ to $t = 3$ for the function $f(t) = 2t^2 - t$ using the above formula.

Rob also correctly computed average rates of change between $t$ and $t + h$ for various functions, including $f(t) = -2t + 3$, $f(t) = t^2$, and $f(t) = t + t^2$ during in-class activities and on his homework assignments. Furthermore, Rob included a discussion about the symbolic representation on his concept map of Rate of Change (see Appendix D). Rob indicated that average rate of change is defined as $\frac{f(x + h) - f(x)}{h}$ and that $\frac{f(x + h) - f(x)}{h}$ is the slope of the secant line between two points.

In addition to talking about the symbolic representation in his concept map of Rate of Change, Rob also included the graphical and physical representations. Rob excluded any discussion about the numeric representation.
A car travels for 30 miles with an average velocity of 40mph and then for another 30 miles with an average velocity of 60mph. \[6, p. 149\]

(a) What is the average velocity of the car for the entire trip?
\[
d = vt
\]
\[
30 = 40t + 60t
\]
\[
t = 75\text{ hr}
\]
\[
30 = 60t
\]
\[
t = 0.5\text{ hr}
\]
\[
\text{average vel} = \frac{90 + 60}{2} = 75 \text{ mph}
\]

(b) Another car travels for 30 minutes at 40mph and then for 30 minutes at 60mph. Find the average velocity over the 1-hour time period.
\[
d = vt
\]
\[
d = 40(0.5) = 20 \text{ miles}
\]
\[
d = 60(0.5) = 30 \text{ miles}
\]
\[
d = 50 \text{ miles}
\]
\[
\text{average vel} = \frac{50 \text{ mph}}{1}
\]

(c) A car is to travel 2 miles. It went the first mile at an average velocity of 30mph. The driver wishes to average 60mph for the entire 2-mile trip. Is this possible? Explain.
\[
\text{1 mile traveled at 30 mph}
\]
\[
\frac{20 + x}{2} = 60
\]
\[
20 + x = 120
\]
\[
x = 100 \text{ mph}
\]

\[
\text{yes, but you would need to be driving faster than the speed limit!}
\]

Figure 10: Rob's Average Velocity Homework Problem

Preliminary Document

University of New Hampshire

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This is consistent with Rob’s work throughout the year. Rob tended to draw graphs when given data in a numeric context. For example, on the following examination question, Rob attempted to draw a graph of $t$ versus $f(t)$ to aid in his solution to the problem.

\[
\frac{1 + \frac{1}{25} - 1}{\frac{1}{25}} = 1.
\]

Rob attempted to answer this question by writing \( \frac{1 + \frac{1}{25} - 1}{\frac{1}{25}} = 1 \). Rob seemed to be trying to fit the data into the average rate of change formula.

For \( f(t) \) the sequence of values of \( h \) approaching zero and the corresponding values of the average rate of change from \( t = 1 \) to \( t = 1 + h \) are given in the following table.

<table>
<thead>
<tr>
<th>( h )</th>
<th>Average Rate of Change of ( f(t) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/5</td>
<td>9.4932</td>
</tr>
<tr>
<td>1/25</td>
<td>8.3110</td>
</tr>
<tr>
<td>1/125</td>
<td>8.0992</td>
</tr>
<tr>
<td>1/625</td>
<td>8.0576</td>
</tr>
<tr>
<td>1/3125</td>
<td>8.0493</td>
</tr>
</tbody>
</table>

Find the average rate of change from \( t = 1 \) to \( t = 1 + 1/25 \) and explain its meaning.

In his answer to a related question that asked if \( f(t) \) was increasing or decreasing at \( t = 1 \), Rob wrote that the function \( f(t) \) is “increasing because \( t \) is positive”. From his graph, it appears that Rob is assuming \( f(t) \) is increasing for \( t > 0 \) and \( f(t) \) is negative for \( t < 0 \).

Average Rate of Change Concept Image: Summary. Rob’s performance on interview tasks, homework assignments, examinations, and in-class activities indicate
that the physical presentation makes up a large part of his concept image of average rate of change. Rob’s discussions as he solved problems during the first interview indicate that the symbolic, numeric, and even graphic mediums often do not become representations for Rob. Rob often did not use the symbolic, numeric, and graphic contexts to express his internal presentations. Rather, he chose to use physical representations, which is indicative of his mental image of average rate of change being largely physical.

Early in the semester, Rob also held onto strong images of average rate of change as the numeric average of rates. Additionally, Rob initially did not connect the concept of slope of the secant line with average rate of change. However, later in the semester, as evidenced by his work during classroom activities and his performance on the final examination, Rob abandoned his notion of average rate of change as a literal average and was able to connect the slope of the secant line to the concept of average rate of change.

**Derivative and Integral.** As described in a previous section, Rob approached many derivative and integral problems using a physical representation. Rob especially tended to impose a physical context on graphical derivative problems. In addition to approaching graphical derivative problems using a physical representation, Rob also talked about a procedure for sketching the graphs:

R: OK. Uh...when taking a derivative, you want to look at any points where there’s a horizontal, well, like a zero slope. ‘Cause that would basically be where, like a f max and a f min would be.
I: OK. And why is that important?
R: Um, whenever there’s an f min or an f max the derivative will equal zero for y, y is equal to zero. So it’ll be on the x-axis.
I: OK.
R: So what I want to do is trace it down, find the point...uh...kind of line it up! OK, so I plotted these two points. And another thing to look at is,
the graph is increasing from 0 to this, at max. And you want to write that in.

Notice that Rob first identified points where the function had a maximum or minimum because ‘the derivative will equal zero for y’. Then Rob mapped the function’s maximum and minimum points to the x-axis on the graph of the derivative. Rob seemed to effectively use the first part of this procedure, mapping maximums and minimums on the function graph to zeros on the derivative graph, but there is little evidence that he used the latter part of his procedure, using the increasing and decreasing properties of the graph, to solve problems.

As discussed previously, Rob’s velocity-acceleration confusion caused him to make mistakes on some homework and examination problems early in the semester. However, as the semester progressed, Rob overcame his misconception and solved many homework, examination, and in-class problems correctly. Consider Rob’s work on the following homework problem from late in the first semester presented in Figure 11. Notice that Rob lines up points where the acceleration and position have a y-value of 0. Rob correctly sketched the graphs on this problem.

Rob has also shown that he is comfortable using Riemann Sums to approximate the area under a curve and seems to grasp the connection between Riemann Sums and the integral. He is able to estimate the area under a curve, as evidenced by a number of homework, in-class activity, and examination problems. Furthermore, on his concept map of integral, Rob mentions both that an integral is the area under the curve and that it is the limit of a Riemann sum. When I asked Rob why he would want to take an anti-derivative such as \( \int 3x \, dx \), he responded that the anti-derivative gives the area under the
curve and then proceeds to explain how he sees the relationship between area under a curve and Riemann sums:

You use it to find, um, the area under the curve between those points. All the integral is...from is, um, doing a Riemann sums. Which is where you just draw little rectangles. If you had a curve you draw rectangles and as the number of rectangles approached infinity that where it turns into...the....integral.

Furthermore, Rob was able to interpret his answer to a definite integral problem as the area under the curve. In the next passage, Rob was working on computing $\int_{1}^{5} 3x^2 \, dx$.

R: So...it’ll be ...that is the area under the curve between points 1 and 5. So if I actually graphed this, um....3x squared...(sketching out graph to illustrate)...the graph would look something like that...from 0 to 5, it’s just this area under here.
I: Oh! OK, so that’s what that number 125 stands for?
R: Yeah, so you would have 125 units squared.

Notice that Rob interprets the problem as asking for the area under the graph of the function $3x^2$ from $x = 1$ to $x = 5$. Rob drew a graph of $3x^2$ and shaded the area under the function from $x =1$ to $x = 5$ to give a pictorial presentation of his answer.

Rob was able to sketch position graphs from velocity plots, estimate the change in momentum from a plot of force versus time, and set up and calculate work integrals. For example, consider Rob’s work on the problem pictured in Figure 12, which appeared on Rob’s Final Examination. Notice that Rob correctly computed an upper estimate for the distance in Part (a) and interpreted the meaning of the area under the velocity versus time plot in Part (b). Rob was also confident using rules and formulas to compute derivatives and anti-derivatives of functions.
Sketch plots of the acceleration and position given the velocity below:

![Acceleration Plot](image1)

![Velocity Plot](image2)

![Position Plot](image3)

Figure 11: Rob's Graphical Homework Problem

Consider Rob's concept map for derivative, located in Appendix E. In the central box, Rob includes the symbols $f'(x)$, $dy/dx$, and $d/dt$ with the word 'derivative'. Notice
also that Rob commits a large portion of the concept map to the rules for differentiating, showing examples for the chain, product, quotient, and power rules. Further evidence of the prominence of formulas and rules in Rob's thinking is found in his responses to interview questions. For example, Rob encountered some difficulty while working on Derivative Task Problem 1.

5. (12 calc pts) A car comes to a stop five seconds after the driver applies the brakes. In the table below the velocity of the car is given for the first three seconds after the brakes have been applied. You may want to plot the function.

<table>
<thead>
<tr>
<th>Time since brakes applied (sec.)</th>
<th>Velocity (ft/sec.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>90</td>
</tr>
<tr>
<td>0.5</td>
<td>60</td>
</tr>
<tr>
<td>1.0</td>
<td>40</td>
</tr>
<tr>
<td>1.5</td>
<td>30</td>
</tr>
<tr>
<td>2.0</td>
<td>22</td>
</tr>
<tr>
<td>2.5</td>
<td>15</td>
</tr>
<tr>
<td>3.0</td>
<td>10</td>
</tr>
</tbody>
</table>

(a) Compute an upper estimate for the distance the car traveled (for the given times) after the brakes were applied.

\[ \text{Upper} = \frac{1}{2}(y_1) + \frac{1}{2}(y_2) + \frac{1}{2}(y_3) + \frac{1}{2}(y_4) + \frac{1}{2}(y_5) + \frac{1}{2}(15) \]

\[ \text{Upper} = \frac{1}{2}(257) \]

\[ \text{Upper} = 128.5 \text{ ft} \]

Figure 12: Rob's Riemann Sums Final Examination Problem

He was confused about where the points that have a horizontal tangent line on the graph of a function map to on the graph of the derivative. Rob also struggled with the significance of the points on the graph of the function that cross the horizontal axis. Rob
was confounding the processes of sketching derivatives and anti-derivatives. I asked Rob if there was any way that he could verify if one of his processes was correct. Rob responded, “If there was an equation I could do it, but graphically it’s the part I’m always looking at. I’m not sure…” This is an indication of Rob’s comfort working with formulas. Further evidence of Rob’s comfort working with formulas was found in his work throughout the year. Rob correctly computed most derivatives and anti-derivatives of polynomials, exponential and logarithmic functions, and trigonometric functions on homework assignments, in-class activities, and examinations.

Although Rob was comfortable working with formulas to compute derivatives and anti-derivatives, Rob sometimes had trouble recognizing or applying formulas in situations he had not previously encountered. For instance, Rob was unable to interpret the expressions in Derivative Tasks 5.2 and 5.3 as approximations of the derivative. He stated that the expression in Task 5.2 “…looks like the definition, which is what the derivative is based on…” but interpreted the question as asking what point on the graph of f(t) is closest to 0.002. Rob’s inability to apply or recognize derivative and anti-derivative formulas in novel situations is an indication that his concept image of derivative and anti-derivative is unbalanced.

Rob also exhibited a weak understanding of certain aspects of the integral, specifically the idea of an integral as a limit of sums. During the third interview, I asked Rob to find the anti-derivative of 3x². He set up an indefinite integral, but had trouble explaining why he needed to write $dx$ next to the $3x^2$.

It...I guess it just, it’s... it’s how you write it. I guess it’s just some rule where you put it there – I don’t know why – you take the anti-derivative. It, it’s just like when you take a derivative you put kind of dy/dx when you just, with respect to dx.
Rob mentions that placing the $dx$ in an integral is some kind of rule that doesn’t have much meaning to him. Rob failed to recognize that $dx$ was derived from the width of an interval of the sum.

Rob had some difficulty solving derivative and integral problems presented to him in a numeric medium. For example, Rob was unable to answer some problems that required him to use the chain rule to compute the derivative of a composition of functions when the function data was presented to him in a table. When solving problems involving a table of values, Rob often drew a graph of the data to help him answer questions.

Finally, although Rob rarely mentioned the Fundamental Theorem of Calculus during any of his interviews, Rob’s discussions during the interviews showed evidence of his understanding of this important theorem. For example, consider the following passage, from Rob’s discussion of Integral Tasks 2 and 3:

I think of this probably as.... well just something you use for doing the physics part of calculus. ‘Cause when you take the anti-derivative... ‘cause you know that, like position, um you take the derivative of this equation, you take the derivative of it and you get the velocity and the derivative of velocity is the acceleration. And it works the same way if you go backwards. The anti-derivative of the acceleration is the velocity. Anti-derivative of this is...position.

The idea that the integral is ‘going backwards’ is precisely what Rob refers to in the physics portion of his concept map of integral. Rob wrote that “The derivative moves one way, the integral moves the other.” It seems that Rob has an intuitive understanding of the Fundamental Theorem of Calculus, even though he rarely mentions it during the interviews.
**Derivative and Integral Concept Image: Summary.** Much like his concept image of average rate of change, the physical presentation made up a large part of Rob’s concept images of derivative and integral. Rob repeatedly spoke about derivative and integral problems using the physical representation, even when the problems were presented to him in a formal, mathematical (non-physical) manner. Rob tended to impose a physical representation most often when he worked with graphical derivative and integral problems. Rob’s concept maps of derivative and integral also offer supporting evidence of the important role of physics concepts in his concept images of derivative and integral. Rob included a physics branch in each of his concept maps of derivative and integral.

**Summary**

Rob was classified as a Contextualizer in the categories of Average Rate of Change and Derivative and Integral. Rob’s concept image was dominated by physical presentation, which were manifested by his use of physical representations as he solved interview tasks, homework and examination problems, and in-class activities. Additionally, Rob’s concept maps of average rate of change, derivative, and integral indicate that physics concepts played an important role in his conceptualization of these calculus concepts.

**Scott**

**Background**

Scott is a physics major who became interested in pursuing physics after touring the Kennedy Space Center. Scott is very interested in space science. During the Fall
semester, in addition to the Calculus/Physics course, Scott enrolled in *General Chemistry* and *Astronomy*. In the Spring semester, Scott was enrolled in the second half of *General Chemistry* and *Introductory English*. Scott indicated that he decided to enroll in Calculus/Physics because he thought that having calculus and physics integrated would benefit him in the long run.

Scott is a member of the Society of Physics Students and likes to spend his free time riding his bike, playing computer games, spending time with his friends, and going to the gym. Scott hopes to get a job in a lab during his Sophomore year in college and would like to obtain a summer internship doing research somewhere outside of New Hampshire. Scott indicated that his long term plans include getting a Ph.D. and going into physics research.

Scott took a yearlong Advanced Placement calculus course in high school. He reported that he covered derivatives and integrals in his high school calculus class and specifically mentioned calculating areas under curves. Scott also took a non-calculus-based physics class in high school. He recalled working with kinematics, centripetal forces, and friction. Scott indicated that he liked working on labs because he got to see why things were happening.

Scott received a B+ in the first semester of calculus. Twenty five out of 48 students in the Calculus/Physics class received a grade in the range of B- to B+. Scott received a B in his first semester of physics.

**Physics Use Classification**

**Overview.** Scott was classified as a Contextualizer in the category of Average Rate of Change and an Example-User in the category of Derivative and Integral. Scott
was classified as a Contextualizer in the category of Average Rate of Change because his internal images of average rate of change were frequently manifested in physical representations. Scott often used physics concepts to describe his presentations of average rate of change.

Unlike the prominent role physics concepts played in his conceptualization of average rate of change, Scott relied less upon physics concepts to aid in his understanding of the derivative and integral concepts. Scott's did not use physics to represent derivative and integral problems as frequently as he used physics to represent average rate of change problems. Scott mentioned examples of physics problems or concepts as he worked through derivative and integral problems. However, there was no indication from either Scott's descriptions of his solution process or his work that he was using strictly a physical representation to solve derivative and integral problems.

The next two sections present evidence for Scott's classifications as a Contextualizer in the category of Average Rate of Change and as an Example-User in the category of Derivative and Integral.

**Average Rate of Change: Contextualizer.** Scott often invoked a physical representation as he solved average rate of change problems. In particular, Scott talked through many of his solutions to average rate of change problems during the interview sessions as if they were given to him in a physics context. Scott's first interview with me took place as the Calculus/Physics class finished working with average and instantaneous rates of change and exploring relationships between position, velocity, and acceleration plots. Scott originally indicated that he had encountered the expressions $\frac{f(x_1) - f(x_0)}{x_1 - x_0}$ and $f(x_1) - f(x_0)$ in high school calculus and physics classes, but was unable to explain
what the expressions meant when he took the Average Rate of Change Pretest on the first
day of class. However, during the first interview, when I asked Scott to tell me what the
above expressions meant, he freely interspersed the phrases ‘average rate of change’ and
‘average velocity’. Talking about the expression \( \frac{f(x_1) - f(x_0)}{x_1 - x_0} \), Scott said:

Um, well, when I first did this, I wasn’t quite sure. It looked very familiar. And, so, but, so I couldn’t decide what it was. But now that I’ve gone through the classes so far, it looks like the formula for average rate of change. The change in distance over the change in time. Or average velocity.

Scott’s language in the above passage seems to indicate that he thinks about average rate
of change and average velocity as the same concept. In particular, Scott talked about the
expression \( \frac{f(x_1) - f(x_0)}{x_1 - x_0} \) meaning change in distance over change in time even though the variables \( x_1, x_0, \) and \( f(x) \) are abstract. Scott’s conception of \( \frac{f(x_1) - f(x_0)}{x_1 - x_0} \) as average velocity was manifested in

Scott used physics to interpret the average rate of change in other places, as well.

For instance, as Scott talked about his solutions to the Average Rate of Change Problems 5 through 8, he indicated that he was thinking about the problems in a physical context.

In particular, as Scott discussed his solution to Average Rate of Change Problem 6, he talked about the graph as if it were a graph of average velocity.

S: I started at \(-1\) and went to 2 and I realized that right here [portion of the
graph between \(-1\) and 1] those would cancel each other out.
I: Ah, between \(-1\) and 1?
S: Right. So, I did that, so… I… that’s what I was thinking. ‘Cause I noticed that it started here but those would cancel each other out for average velocity so it would be that part of the function [between 1 and 2].
I: OK. So you just calculated it between 1 and 2?
S: Right.
Scott reasoned that the portion of the graph from \( x = -1 \) to \( x = 1 \) could be ignored since, if it is a position versus time graph, the object has traveled toward a point and then back to its original position – thus starting and ending in the same place. Scott indicated that thinking about the graph in this way justifies ignoring the portion of the graph from \((-1, 1)\) to \((1, 1)\). Scott’s physical representation of the problem seemed to distract him from noticing the conflict between his answer and slope of the secant line joining \((-1, 1)\) and \((2, 4)\). Scott failed to notice that the slope of the secant line connecting \((-1, 1)\) and \((2, 4)\) was different from the slope of the secant line connecting \((1, 1)\) and \((2, 4)\). Scott’s consideration of physical experiences appears to dominate his thinking about this problem.

Scott’s answers to the Average Rate of Change interview tasks, along with his in-class work and examinations indicate that Scott’s presentation of average rate of change was largely physical. Scott used the physical representation to solve many average rate of change problems. A more extensive discussion of Scott’s conceptualization of average rate of change is presented in a forthcoming section.

**Derivative and Integral: Example-User.** While Scott interpreted many average rate of change problems as average velocity problems, I did not observe him making similar physical interpretations of derivative and integral problems. Rather, Scott tended to evoke physics problems or concepts in his discussion of solutions to the interview tasks. The most striking example happened after Scott completed Derivative Task 2. He was talking through his solution process and stated that he thought about certain pieces of the graph in terms of motion:

I know a couple of times, like here, I was trying not to say acceleration or velocity. Sometimes when I see those, I tend to look like, okay, straight
line...I tend to see like acceleration or position or something like that...Well, right here, I mean, just because there was a straight line, like I assumed, I just thought to say constant acceleration or something like that. But just, just how it looked, looked like acceleration. But because this whole graph would be acceleration because it -- well, but actually...it's...position. But I just, just to look at the graph, it's just kind of in there.

Scott claimed that when he saw the straight line portion of the graph, he was reminded of a graph of constant acceleration. However, there is no indication that Scott used the concept of constant acceleration to solve the above problem. In fact, if Scott used constant acceleration to help him solve the problem, he would have sketched a graph of the jerk of the object. Scott never mentioned the concept of jerk either during the interviews or in his concept map of Derivative and Integral (see Appendix E).

Scott included the physics concepts of position, velocity, acceleration, flux, and center of mass in his concept map of Derivative and Integral. Scott indicated with arrows how the acceleration, velocity, and position are related and connected the three concepts to the central concept of derivative. Although he included these physics concepts in his concept map of derivative, Scott explicitly stated in one interview that he thought about calculus and physics separately.

I mean, I usually think, I usually kind of keep separate in my head, like acceleration goes with position and you know, concavity goes, I mean, f double prime and f (t). So usually I'm thinking physics and math separate.

Scott explicitly stated that he thought about calculus and physics concepts separately. However, Scott’s inclusion of physics concepts in his concept map of derivative and integral and his tendency to mention physics examples as he solved derivative and integral problems indicate that there existed an overlap in Scott’s concept images of derivative, integral, and certain physics concepts.
Summary. Scott solved many average rate of change problems by invoking a physics representation. Scott’s extensive use of the physical representation indicates that his concept image of average rate of change was largely made up of a physical presentation. Scott relied less on the physical representation to solve derivative and integral problems. Scott was comfortable using mathematical concepts to solve derivative and integral problems, as will be seen in a forthcoming section. In the next section, I discuss my interpretation of Scott’s concept images of average rate of change, derivative, and integral.

Concept Image

Overview. In this section, I will discuss Scott’s concept images of average rate of change, derivative, and integral. I attempted to re-construct Scott’s concept images by using his concept maps as well as his responses to interview tasks, homework assignments, examination questions, and in-class activities.

Average Rate of Change. As described in a previous section, Scott’s approached many average rate of change problems using a physical representation. Recall that Scott often talked about average rate of change as if it were average velocity, interpreting $f(x_1) - f(x_0)$ as change in distance and $x_1 - x_0$ as change in time. However, Scott’s discussions during the first interview indicate that the symbolic, numeric, and graphic mediums become representations for Scott. Scott often used these mediums to express his internal presentations. Scott also appropriately connected the idea of average rate of change with slope of the secant line.

Scott indicated that he associated slope with rate of change on his concept map for rate of change (see Appendix D). Scott’s association of slope and average rate of change
on his concept map is consistent with his work on homework problems, in-class activity problems, examination questions, and interview tasks. For example, Scott’s response to Average Rate of Change Problem 5 is a strong indication of his thinking about the relationship between the slope of the secant line and the average rate of change:

“Umm... well the secant line is the average rate of change through two points... an... so secant line between here would be, represent the average of change.”

Scott also talked about the slope of the secant line as he solved Average Rate of Change Problem 7.

Well, it’s asking for the average rate of change between -3 and 3, so it would be up here [denoting (-3, 9) and (3, 9)]. And the secant line is horizontal, which slope would be zero. And by looking at the graph, that’s how I got that.

Notice that Scott relied on his knowledge that the slope of a horizontal secant line is zero to solve Average Rate of Change Problem 7.

Scott also used the slope of the secant line to justify some of his answers to other Average Rate of Change Problems. For instance, when I asked him to talk about how he made sense of his answers to Average Rate of Change Problem 5, Scott talked about the steepness of the slope of the secant line.

S: Ummm, well... ‘Cause it’s not like too extreme. This, it... chang... (pause) ‘cause I was, I was think -- the secant line going through here. It’s sort of steep, but it’s not incredibly steep, so it wouldn’t be a large number. And it’s not flat, so it wouldn’t be real small. So it would be somewhere in the middle, I guess.
I: OK. And what does secant line have to do with everything?
S: Umm... well the secant line is the average rate of change through two points... an... so secant line between here would be, represent the average of change.
Scott specifically states that the secant line is the average rate of change through two points. It is clear from this passage that Scott has a well-developed conceptualization of average rate of change as the slope of the secant line.

As he worked through the calculus in-class activities, Scott used secant lines to solve average rate of change problems presented in graphical contexts. Scott also used secant lines to help him solve problems asking him to find the average rate of change between \( t = 1 \) and \( t = 1 + h \) of familiar functions, such as \( f(x) = x^2 \). For example, consider Scott’s solution to the following problem shown in Figure 13.

2. Find the average rate of change of \( g(t) = t^2 \) from \( t = 1 \) to \( t = 1 + h \).

![Figure 13: Scott’s solution to an Average Rate of Change Homework Problem](image)

Scott sketched the parabola \( g(t) = t^2 \), sketched a secant line connecting the points \((1,1)\) and \((h + 1, (h + 1)^2)\), and used a triangle to help him solve the problem. Figure 13 is another example demonstrating Scott’s understanding of average rate of change as the slope of the secant line.

Interestingly, Scott did not talk about the slope of the secant line when problems were presented to him in physical contexts. For example, as Scott solved the Average Rate of Change Problem 10, which asked him to find the average velocity of a car between two different times, he talked about the average velocity in terms of change in
distance over change in time, even though the data was presented to him in a graphical context.

OK. So the time of 1, it looks approximately about... I'd say... .7... .75... .6. So at time 1 it looks like it is about .6. So I... for f(1) it equals .6. And then at 3, it looks like it's about -.5. So f at 3 equals -.5. So I'd write -.5 minus .6 over 3 minus 1. Again, change in distance over the change in time. And I'd get... 1.1 over 2. I'd get... the average velocity.

The language Scott used to solve this problem is different from the language he previously used to solve other average rate of change problems, specifically Average Rate of Change Problems 5 through 8. Scott specifically mentioned the formula for average velocity, change in distance divided by change in time. There was no evidence that Scott was using or thinking about slope to solve this problem.

Scott demonstrated his ability to solve symbolic average rate of change problems on a number of occasions. As he answered Average Rate of Change Problem 5, Scott talks about using the expression \( \frac{f(x) - f(x_0)}{x_1 - x_0} \) to solve the problem.

OK. Um, for this one [5], I used that formula that was on the other side. Ah, \( \frac{f(x) - f(x_0)}{x_1 - x_0} \). And that's 4, 4 minus 1 and then over 1, yeah. I may have... I also might... I was recognized this as the function x, y = x^2.

Here, Scott not only explicitly talked about his use of the expression \( \frac{f(x_1) - f(x_0)}{x_1 - x_0} \) in his solution of the problem, but he also pointed out that he recognized the graph as the function y = x^2. Scott did not give any further indication of his use of the formula y = x^2 to help him solve the problems.

Scott easily solved average rate of change problems that involved data presented to him in tabular or numeric contexts. Scott was able to calculate the average rate of
change or average velocity of an object from data presented in a position versus time table on in-class activities, interview tasks, and homework problems. Additionally Scott correctly computed change in distance from a table of time and average velocity values on an in-class activity.

**Average Rate of Change Concept Image: Summary.** Scott often referred to average rate of change as average velocity. When Scott was presented with the expression \( \frac{f(x) - f(x_0)}{x - x_0} \), he identified it as the formula for average velocity and claimed that the numerator represented the change in distance while the denominator represented the change in time. Thus, Scott seems to submerge abstract mathematical formulas and expressions for average rate of change in a physical context in order to make sense of the abstract mathematical notation.

Scott’s tendency to submerge average rate of change problems in a physical context sometimes conflicted with his notion of average rate of change as the slope of a secant line. As evidenced by his work on some of the Average Rate of Change interview tasks, Scott’s interpretation of the problems in a physical way caused him to ignore certain properties of the slope of the secant line. In particular, Scott did not recognize the difference in the steepness of the slopes of the secant lines connecting the points (-1, 1) to (2, 4) and (1, 1) to (2, 4) as he solved Average Rate of Change Problem 6. Instead, Scott’s notion of average rate of change as average velocity seemed to dominate his thinking as he “canceled out” the portion of the graph from \( x = -1 \) to \( x = 1 \) and considered only the secant line from (1, 1) to (2, 4) as he solved the problem.

Scott proficiently worked with average rate of change problems presented to him in graphical, physical, numeric, and symbolic contexts. Scott’s concept map of rate of
change indicates that he thinks about rate of change in three main ways: (1) As a slope, (2) As a change, and (3) As a difference. Scott’s homework, in-class work, examinations, and his performance on interview tasks indicate that Scott not only mentally relates the ideas of slope, change, and difference to rate of change, but that the relationships between slope, change, difference, and rate of change are manifested in his work.

**Derivative and Integral.** As described in a previous section, Scott sometimes invoked examples from physics as he talked through his solutions to derivative and integral problems. Scott included many physics examples on his concept map of derivative and integral. Scott did not explicitly mention properties of graphs on his concept map of derivative and integral, but I noticed that he tended to follow a procedure when graphing derivatives and anti-derivatives. When he graphed the derivative of a function, Scott first located points on the graph of the function where the slope of the tangent line was horizontal and plotted those points on the horizontal axis of the derivative graph. Then he examined what the graph of the function was doing in between the points where the slope of the tangent line is zero and filled in the graph of the derivative. In the following passage, which is representative of the manner in which Scott solved graphical derivative and anti-derivative problems, Scott was working on Derivative Task 1.

Okay. We'll look at the zeroes or the places where there will be horizontal tangents. So of course I just look at those first. So at those points, look over, it's about there, I marked those points as zero. Because that's, the max there is zero and f prime is the slope at that. So at that point it would be zero. So I have places where it would be zero, marked off here. So then the graph is increasing here to that point. So it would be positive until whatever point that is at. So it would be positive, and then it would intersect the x axis, where it's decreasing to the next point. So go down
and then go back up to meet that other zero right here, where it is increasing again. Like that, increasing, or if it’s increasing it's positive.

Notice that Scott immediately marked the points where the slope of the tangent was zero and then he evaluated the shape of the graph in between those points. Scott used the analogy of a number line to describe his method of graphing derivative functions:

I: So I hear you say a lot of, this is where it's increasing and decreasing. So is that primarily what you think about when you go to graph a derivative of a function? Is that like what you’re looking at?
S: Yeah, well when I'm looking at the graph, that's what I'm looking at. Because I sort of, just, instead of just looking at all those wavies and like places where it's turning like that, I just like to break it down to zeroes, and then increasing or decreasing. So just like turn the whole graph into like a number line where I have zeroes, and positive, negative and positive, so you can break down and look at it like that.

Scott’s reason for first locating the zeroes and horizontal tangent lines on a graph is that, “Those are the only real points you can sometimes be sure about when you just have the graph. You usually can only be sure about zeroes and horizontal tangents.”

When Scott sketches the graph of an anti-derivative, he uses mainly the same procedure for sketching derivatives, only in reverse. He first looks at points on the graph of the derivative function that cross the horizontal axis and notes that those are places where the function, or the anti-derivative, changes direction. Then he uses his number line analogy to sketch in the rest of the graph. Interestingly, Scott uses notions of concavity to help him graph the anti-derivative function. Scott seemed to be the only student to make use of concavity in this way; most of the other student avoided concavity or made comments that they did not fully grasp the idea of concavity.

Well, when I look at g prime, I look for places where it's zero, (on the axis), so it looks like it's zero there, and it's zero there. And at these points there would be horizontal tangents. Because if the derivative is zero then it's a horizontal tangent at that point. So somewhere on this line or this vertical line is a tangent, and same here. And then because this is positive,
it's increasing to this, it's increasing. And it's, and this function is
decreasing, so it's concave down. So, to that point. Then it looks like it
changes there, its concavity. But it's -- going down here, it's negative. So
it's decreasing. And it looks like it's decreasing to here where it switches
to increasing. So it would be, decreasing, (would be) concave down. And
then switching to concave up. Looks like switching to concave up right
about here. Change in concavity. And then it's positive. So it's increasing
again. And it's increasing its positive. So it's increasing or, increasing
concave up, positive.

Scott’s facility working with the concept of concavity could contribute to his rich
conceptualization of the relationship between a function and its derivative.

Scott demonstrated his knowledge of the relationship between a function and its
derivative during class discussions, in-class activities, homework assignments, and
examinations. For example, Scott correctly identified the functions in the following
problem, pictured in Figure 14, that was given to him on an in-class activity. Notice that
Scott uses the relationship between a function and its first and second derivatives to
justify his answers. Furthermore, when the calculus class began exploring the
relationship between features of the graph of a function and its derivative in late
September, Scott demonstrated his understanding of the relationship between a function
and its derivative. During a calculus class discussion, the calculus instructor asked the
class, “When f(t) is increasing, what’s f′(t) doing?” After a long pause, Scott answered,
“When f(t) is increasing, f′(t) is positive.”

In addition to his rich conceptualization of the relationship between a function and
its derivative, and the relationship between a derivative and integral, Scott also grasped
the idea that the integral represented the area under a curve. When presented with
velocity versus time graphs of an object, Scott estimated the distance the object traveled
over various time intervals by computing Riemann sums. Scott also used the area under
the graph of a force versus time function to compute the change in momentum.

4. The graphs [(i), (ii), and (iii) given below] are the graphs of a function $f$ and its
first two derivatives $f'$ and $f''$ (though not necessarily in that order). Identify which
of these graphs is the graph of $f$, which is the graph of $f'$ and which is the graph
of $f''$. Justify your answers. [2, p. 62]

![Graphs of function and derivatives](image)

Figure 14: Scott's solution to an in-class activity

When I ask Scott why someone would want to find an indefinite integral, he says,
"If you have a function and you want to know what its – or the derivative or something,
and you have a graph of it, you can use it to check your graph." Scott indicated that the
main difference he saw between $\int_0^5 3x^2 \, dx$ and $\int_0^5 3x^2 \, dx$ was that the former yields a function as an answer, it is more "general" and the latter yields an area under the curve, it is more "specific".

Scott answered many symbolic derivative and integral problems correctly on examinations, homework assignments, and in-class activities. Scott competently applied the power rule, product rule, quotient rule, and chain rule to compute derivatives of polynomials, exponential and logarithmic functions, and trigonometric functions. The symbolic integral tasks involved the substitution method of solution with polynomial functions, exponential and logarithmic functions, trigonometric functions, and inverse trigonometric functions. Scott's ability to work with derivative and integral tasks in the symbolic representation was also manifested in his concept map of derivative and integral (see Appendix E). Scott's concept map of derivative and integral included formulas for taking derivatives and integrals, as well as common notation associated with derivatives and integrals, such as $dt$ and $\Delta x$. During an interview, Scott indicated that when he hears the word "integral" he thinks about "the integral sign" and "the reverse of derivatives".

Scott's use of the physical representation was also evident in problems that included physics contexts or concepts in their statements. Consider Scott’s solutions to a problem on the first examination that asked him to sketch graphs of a student’s motion and explain the graphs. Scott’s solution appears in Figure 15. Notice that Scott used a combination of calculus and physics concepts to solve the problem. In particular, Scott justifies his negative acceleration graph by explaining that the "velocity is decreasing to 0 then increasing in a negative direction."

A student walks beside a 2-meter measuring stick, beginning her walk at the origin. Then she moves with decreasing speed toward the 2 meter mark. After coming momentarily to rest near the 2 meter mark, the student immediately begins moving toward the 0 meter mark with increasing speed. For each of the plots below, sketch graphs of this motion and briefly explain why you drew the plots as you did.

**Position vs. time**

- **Brief Explanation:**
  - She slows as she reaches 2m and stops.
  - Then she changes direction from positive to negative.

**Velocity vs. time**

- **Brief Explanation:**
  - Because her velocity is decreasing to zero and becomes negative.

**Acceleration vs. time**

- **Brief Explanation:**
  - Acceleration is negative because the velocity is decreasing to zero.
Additionally, Scott’s response to Interview 4 Question 2 indicated that he relied heavily on the physical context of the problem help him answer the question.

Well, I know the acceleration is going to kinda look like... either a sine or a cosine. Because the force is going to be, it’s going to oscillate. It’s going to go, up... er, the acceleration is going to go up to zero — or up, yeah, it’s going to decrease to zero and then it’s going to increase and then it’s going to decrease... so it’s gonna, it’s gonna keep doing that...

Scott used his knowledge of the spring force to initially determine the graph of the acceleration. Scott knew that the graph was either a sine or cosine function. Later, Scott said, “It’s just that basically because I know springs, that it’s going to oscillate, and because there’s a restoring force it’s going to... it’s going to keep oscillating.”

Scott was able to find the velocity and position of an object given its acceleration by taking anti-derivatives and using initial conditions. Scott correctly answered the following questions which appeared on his first examination.

You visit the Little Prince on his planet, and find to your amazement that when you drop a ball, it experiences an acceleration that changes cubically in time: \( a(t) = -3.2 \frac{m}{s^2} t^3 \) where the acceleration is downward, toward the center of the planet and \( t = 0 \) seconds when you let go of the ball. If you toss a ball with an initial speed of 8m/s upward, and an initial height of 3m

(a) Find \( v(t) \) for this ball.

(b) Find \( y(t) \) for this ball.

(c) When will it reach its highest point?

Scott used anti-derivatives and the initial conditions to correctly answer parts (a) and (b) of the examination question. Scott used his knowledge that the velocity of the ball is zero when the position of the ball is a maximum to solve part (c).
Scott demonstrated proficiency solving derivative and integral problems presented to him in a numeric context on examinations, homework assignments, and in-class activities. For example, Scott correctly estimated the derivative of \( f(1) \) from a table of average rate of change values for the function \( f(t) \) over different time intervals, as indicated in the problem below. Working with the same function, \( f(t) \) presented below, Scott correctly determined that the function was increasing at \( t = 1 \). Additionally, Scott competently applied the chain rule to compute derivatives of various compositions of functions when the function values were presented in a table.

For \( f(t) \) the sequence of values of \( h \) approaching zero and the corresponding values of the average rate of change from \( t = 1 \) to \( t = 1 + h \) are given in the following table.

<table>
<thead>
<tr>
<th>( h )</th>
<th>Average Rate of Change of ( f(t) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/5</td>
<td>9.4932</td>
</tr>
<tr>
<td>1/25</td>
<td>8.3110</td>
</tr>
<tr>
<td>1/125</td>
<td>8.0992</td>
</tr>
<tr>
<td>1/625</td>
<td>8.0576</td>
</tr>
<tr>
<td>1/3125</td>
<td>8.0493</td>
</tr>
</tbody>
</table>

Scott also competently worked with integral problems in a numeric representation. For example, on examinations, homework assignments, and in-class activities, Scott computed lower and upper estimates for the distance an object traveled from a velocity versus time table. When solving problems involving a table of values, Scott often drew a graph of the data to help him answer questions.

Finally, Scott made a connection between the concepts of derivative and integral through his understanding of the Fundamental Theorem of Calculus. Scott demonstrated
his knowledge of the Fundamental Theorem of Calculus on interview tasks, homework assignments, in-class activities, and examinations. For example, on Integral Task 1, Scott correctly identified the maximum points for the function F(x) at t = 1, t = 5, and t = 9. Scott justified his answer by saying that, "...f(t) is the derivative of F(x) so from 0 to 10 it [the maximums] would be the points where there were zeros. It would be the points were there could be maxes or mins." Scott indicated in the third interview that he thinks of the integral as the "reverse of the derivative." Scott seemed to associate the definition of the integral with the Fundamental Theorem of Calculus. On an examination question that asked Scott to state the definition of the integral, he wrote the following:

\[
\int_{a}^{b} f'(x) \, dx = f(x) \big|_{a}^{b} = f(b) - f(a) \quad \text{and} \quad \int f'(x) \, dx = f(x) + C.
\]

Furthermore, Scott used the Fundamental Theorem of Calculus to connect the derivative with the integral on his concept map of integral.

**Derivative and Integral Concept Images: Summary.** Scott appeared comfortable working with derivatives and integrals in a variety of representations. Scott’s concept image of both derivative and integral seem to be balanced; that is, Scott worked proficiently with the symbolic, numeric, graphical, and physical representations of derivative and integral. Scott’s concept map of derivative and integral, while focused on the symbolic representation, also show the prominence of examples in Scott’s conceptualization of derivative and integral. Scott included physics examples in his concept map of derivative and integral, namely the relationship between position, velocity, and acceleration, work, flux, and center of mass.
Summary

Scott was classified as a Contextualizer in the category of Average Rate of Change and an Example-User in the category of Derivative and Integral. Scott’s concept images of average rate of change, derivative, and integral were balanced; that is Scott showed evidence of his competence working with these calculus concepts in a variety of representations. Scott’s concept map of derivative and integral was dominated by symbols and physics examples, corroborating evidence that physics examples were an essential part of Scott’s conceptualization of derivative and integral.

Background

Terry is an Electrical Engineering major who chose to study in electrical engineering because he likes taking apart and fixing electrical objects such as VCRs and clocks. Terry also likes working with computers. Terry hopes to obtain a summer internship or possibly work in the IOL laboratory next year.

Terry indicated that he enrolled in the Calculus/Physics class because of his involvement in the honors program. In the fall semester, Terry enrolled in Introduction to Electrical Engineering and an Honors seminar that focused on the value of higher education, in addition to the Calculus/Physics course. Terry enrolled in Introduction to English and Introduction to Scientific Programming in addition to Calculus/Physics during the spring semester.
Terry was involved in intramural soccer throughout the year and reported that he enjoys reading and writing science fiction in his free time. Terry indicated that he tends to spend a lot of his time studying. Terry worked as an assistant manager at a local grocery store in his hometown during the summers and on school breaks.

Terry reported that he took a year-long AP calculus class in high school but that he didn’t do well in the class. Terry’s calculus class covered topics such as solids of revolution, derivatives of major functions, integrals of major functions, and area under curves. Terry reported that he did not like working with solids of revolution. Terry took physics during his junior year in high school. Terry reported that he did well in his physics class and particularly liked the teacher of this class. Terry’s physics class covered such topics as motion including circular motion, electricity and magnetism, circuits, optics, and thermodynamics. Terry reported that he particularly enjoyed working with waves during lessons on electricity and magnetism. Part of Terry’s physics class involved semester long projects in which the entire class worked together on a project that incorporated every topic that they covered in the physics curriculum. Terry’s class built a house that demonstrated the various concepts. He worked on a part of the house involving the circuitry. Terry also lived abroad while growing up since his parents were in the military.

Terry received an A in the fall semester of calculus. Thirteen out of 48 students received an A- or an A in calculus during the fall semester. Terry also received an A in his first semester of physics.
Physics Use Classification

Overview. Terry was classified as a Language-Mixer in the categories of Average Rate of Change and Derivative and Integral. Terry was classified as a Language-Mixer in both categories because he frequently interspersed physics terminology with mathematical vocabulary as he worked through calculus problems. Terry’s internal presentations of average rate of change, derivative, and integral appeared to be balanced; that is, Terry seemed to competently work with calculus problems in multiple representations and did not necessarily show a preference for working in one representation over another.

The next two sections present evidence for Terry’s classification as a Language-Mixer in the categories of Average Rate of Change and Derivative and Integral.

Average Rate of Change: Language-Mixer. The physical representation played an important, but not dominant, role in Terry’s conceptualization of average rate of change. Terry denoted “motion” and “implies something/time, usually rate” as two direct links to the central concept of Rate of Change in his concept map of rate of change (see Appendix D). His concept map of rate of change is filled with examples from physics and everyday life and he even includes a spoke labeled “Layman’s Terms” which he defines as “How quickly something changes”. Terry’s focus on language in his concept map is consistent with his careful choice of vocabulary as he talked through his problem solutions. An example of the way Terry infused mathematical language and physics terminology as he solved average rate of change problems is found in his solution to Average Rate of Change Problem 5.

Uh, between $x_1$ and $x_2$. Their position, so that’s, position (1, 1) and at $x_2$ it’s (2, 4) and if I remember right, it’s…it’s rise over run. And, so it
should be 4 minus 1 over 2 minus 1, which would be 3 over 1. So it would be 3, I think!

Notice Terry's use of the word position, taken from his physics vocabulary and his use of the mathematical 'rise over run'. Terry interchanged calculus and physics vocabulary as he solved this problem. Later, when I asked Terry to solve Average Rate of Change Problem 7, he immediately realized that the slope of the secant line was horizontal, so the average rate of change was zero. However, he justified his answer by talking about zero displacement: “Cause it’s ZERO! It means it’s a horizontal secant line, which...it didn’t move. Well, it did move, but...in the end it’s displacement was zero.” Here, Terry again interspersed vocabulary from calculus, namely the horizontal secant line, and terminology from physics, namely the displacement. When I prompted Terry to discuss how he made sense of an average rate of change of zero, he used an example from physics to justify his answer:

It, it moved, and there was a period of time that elapsed. But, the position initial was the exact same thing as your position that you observed and so if you ignore everything in between, which is what you’re doing with an average rate of change, it hadn’t moved. If you took a picture of it when it started and then it moved, and then you took a picture of it again right at the time that you were observing, it would have looked like it never had moved.

Notice that Terry used both formal physics vocabulary (position initial, observed position) and an informal discussion about taking a picture of the object.

Although Terry integrated both physics and calculus vocabulary when he solved average rate of change problems, he did not exhibit evidence of submerging problems in physical contexts or working exclusively with the physical representation of average rate of change problems. Similar to Scott’s explicit distinction between calculus and physics problems dealing with derivatives, Terry made a distinction between his physical, or real-
Terry’s response to the Average Rate of Change Problem 1.

First thing I had down was flow. Like flow rate through a tube—like wires and liquids and that kind of thing. And then the second one I had down was calculus because I remember doing rate of change last year...and that was just, that was really the only thing that came to mind then. And how fast something changes— we say rate. Rate of something like that. (pause) Um, currency rates—like, how much it, you have to pay to get like to go from one currency to another—is that rate. So that’s what came to mind—from living in Germany thing—currency rates. Always having to deal with that! It wasn’t fun at all! (both laugh)

Terry explicitly noted a distinction between rate as a flow (physics) and rate as a formal, calculus concept. Notice that Terry’s life experiences, dealing with currency exchange rates and working with flow also influenced his thinking about rate. These distinctions (rate in physics, calculus, and real-life) are also evident in his concept map of rate of change (Appendix D). Terry divides his concept map of rate of change into physics applications (motion and flow rate), calculus applications (related rates and derivative) and real-life applications (Layman’s Terms).

Terry went on to explain that even though he worked with rate of change in both calculus and physics, his work with rate of change in physics was more meaningful to him.

I remember it more formally in calculus, but as for applicability, I remember doing more in physics. It just—it seemed like it was textbook problems in calc and then in physics we were actually, you know, this is how you can use it and this is what it does. Which made more sense to me.

Terry also made an interesting observation about the variables in the Average Rate of Change Problems 3 and 4.
So I remember it was in a calc book and really, it just....Actually \( f(x_0) \), like, \( x_0 \) was always a physics term for me. I never used that in calculus, never saw that in calculus. It was like the position initial of something that you took was \( x_0 \). And, so I remember it in that kind of context. So you had your, it wasn’t really an observed position it was just where you were starting your experiment.

Again, this distinction between calculus and physics uses of average rate of change is consistent with Terry’s concept map of average rate of change.

**Derivative and Integral: Language-Mixer.** Terry tended to use both calculus and physics vocabulary as he solved derivative and integral problems. Consider the following excerpt from Terry's second interview in which he was working on Derivative Task 4.

It looks to be around, in here, it's got a positive slope and it's zero, and it goes to another positive slope, so it's probably something in the neighborhood of this. Yeah. And it goes faster and slower and, somewhere around in...I'm killing it.

Terry combined mathematical vocabulary and physics terminology as he solved the above problem. Terry first talked about the slope of the graph and finished his discussion using the physical descriptions "faster and slower" to describe the motion.

In a discussion about anti-derivatives, Terry stated that he thinks about derivatives and integrals as "physics tools". I asked Terry when it would be useful to take an anti-derivative such as \( \int 3x^2 \, dx \). He stated:

If you’re given acceleration/velocity equations like that, you can find position easily. Working backwards, when you take the derivative of position, you get velocity. More physics tools!

Here Terry associated differentiation and integration with kinematics.

However, Terry did not seem to associate physical phenomena as much with derivatives and integrals as he did with the concept of average rate of change. Recall that
Terry included a number of physics concepts and physical examples in his concept map of rate of change. Terry does not mention any physical concepts or examples in his concept map of derivative and integral. A more extensive discussion of Terry's conceptualization of derivative and integral is presented in a forthcoming section.

**Summary.** Terry’s discussions of most average rate of change, derivative, and integral problems involved the mixing of both mathematical and physical vocabulary. As Terry solved average rate of change problems during the interviews, he often incorporated both mathematics and physics terminology into his discussions of the problems. Terry often used phrases such as “initial position” and “final position” when talking through average rates of change problems, even if the problems were not presented in a physical context. Terry did not necessarily submerge the average rate of change problems in a physical context, rather he used the physics language to help him describe the problem situation and his solution. Although Terry used both mathematical and physical language to describe his solution processes, he did not show evidence of that his concept images of average rate of change, derivative, and integral were dominated by physical presentations. In the next section, I discuss my interpretation of Terry’s concept images of average rate of change, derivative, and integral.

**Concept Image**

**Overview.** In this section, I will discuss Terry’s concept images of average rate of change, derivative, and integral. I attempted to re-construct Terry’s concept images by using his concept maps as well as his responses to interview tasks, homework assignments, examinations, and in-class activities.
Average Rate of Change. Terry exhibited evidence of a balanced concept image of average rate of change. Terry comfortably and competently solved average rate of change problems in a variety of representations. For instance, Terry demonstrated his knowledge of the connections between the slope of the secant line and the average rate of change between two points on a number of occasions. Terry talked about the slope of the secant line as he solved many of the Average Rate of Change interview problems. For example, as Terry solved Average Rate of Change Problem 5, he talked about the secant line in order to make sense of his answer:

T: ...It [his answer] would make sense, because that's the slope of the secant line between them.
I: OK. And what does the secant line have to do with average rate of change?
T: (pause) Ah, it's just, the secant line is...it's how, it's...with, with ignoring every point in between the two points you observed, where it went and how long it took it... to get there. Just jumping from point A to point B and not worrying about whether it went north or south from there or east or west, it just ended up there starting from there.

Notice that Terry not only talks about the relationship between the slope of the secant line and the average rate of change but he also infuses physics language into his discussion. Terry describes the motion of an object and ‘how long it took’ for the object to travel between two points. Terry uses this physical language within his discussion of the importance of the secant line. Terry’s recognition of the relationship between the slope of the secant line and the average rate of change and his ability to talk about that relationship is indicative of his conceptualization of average rate of change as the slope of the secant line. Terry also talked about the slope of the secant line as he began to solve Average Rate of Change Problem 7.

Just cause, from looking at it, it's, what's that, 3...way up here, 9. That's (3, 9), that would be the slope of the secant, so that's going to be zero.
Terry also easily explained the relationship between the formula for average rate of change and the slope of the secant line in his answer to Average Rate of Change Problem 8:

You've got the function $x$, or the $f(x)$. And you're looking for, I do it, I'm still visual, I remember it in terms of slope. And rise over run. And the function value is your... is your rise, I always remember that. And so it would be the $f$, like, the $f(x)...$ the, I don't know, $f(x_2)$ or $f(x_1)$ and it seems to me like $b$ would come after 2, like it would be your second point taken, so... you're average rate of change would be the, the $f(b)$ minus the $f(a)$ divided by your, your $x$-values.
Terry competently solved average rate of change problems presented to him in a physical context on homework assignments, examinations, and in-class activities. For example, consider Terry’s solution to the following homework problem, shown in Figure 16. In the above problem, Terry effectively applied his knowledge that the average velocity is equal to the change in position over the change in time. Terry proficiently worked with average rate of change problems in a physical context on interview tasks, homework assignments, and in-class activities.

Terry frequently demonstrated his ability to work with average rate of change problems in a symbolic context. Terry correctly computed the average rate of change between $t = 1$ and $t = 1 + h$ of $f(t) = t$, $g(t) = t^2$, and $h(t) = t + t^2$ on his in-class calculus activities. Terry also correctly computed the average rate of change of $f(t) = 2t^2 - t$ from $t = 1$ to $t = 3$ on his final examination.

Often when he was working though problems with data given to him in a tabular form, Terry made reference to the graphical connections between rate of change and slope. Consider Terry’s solution to Rate of Change Pretest problem 9.1:

The first one find the average velocity of $0 \leq t \leq 0.2$. And...the first set...would then be, you, your set of values for $t$ would be 0 and .2. And then your function values would be given to you on a graph and that’s in 0 feet and...0.5 feet. And so, I’ll do the same thing I did on the other page, which is slope, ‘cause that’s your function, that’s your function value and that’s your, that’s your independent...mmm...yeah, you just get the function values, subtract them.

Although Terry did not sketch a graph to accompany his work, he refers to his work on Average Rate of Change Problems 5-8, problems presented in a graphical context.

Furthermore, Terry used the terminology 'slope' to describe his method of solution for this problem.
**Average Rate of Change Concept Image: Summary.** As Terry solved average rate of change problems during the interviews, he often incorporated both mathematics and physics terminology into his discussions of the problems. Terry often used phrases such as “initial position” and “final position” when talking through average rates of change problems, even if the problems were not presented in a physical context. Terry did not necessarily submerge the average rate of change problems in a physical context, rather he used the physics language to help him describe the problem situation and his solution.

Terry seemed confident working on most problems involving average rate of change. He indicated several times during the interviews that he prefers working in physics contexts because it is easier for him to visualize the physical phenomena versus working with abstract calculus.

On his concept map, Terry connects “slope of tangent” to “rate of change” with the phrase “other names”. This is an indication that Terry associates the physical and graphical contexts of rate of change. Furthermore, his work on interview tasks, homework and examination problems, and in-class activities have shown that he is adept at working with rate of change in a symbolic context.

**Derivative and Integral.** As described in previous sections, Terry interspersed mathematical and physical language as he solved derivative and integral problems. However, the physical representation did not dominate Terry’s conceptualization of derivative and integral. Terry’s concept images of derivative and integral seemed to be balanced; that is, Terry’s presentation of derivative and integral were manifested in multiple representations. Furthermore, Terry did not exhibit a preference for working with one representation over another.
Terry, like many of the other students, talked about a process that he followed when sketching a graph of a derivative or anti-derivative from the graph of a function. Terry described his process as follows:

T: The zeros, which are the tangents, where it will be zero on the original function.... that's where you'll get either a maximum or a minimum, if it goes from increasing to decreasing or vice-versa. So you map out these points.
I: And then, after you map out those points, then what's next in line?
T: Then, leading up to those points, whether it's a positive or a negative tangent, and then draw it accordingly.

Notice that Terry emphasized the importance of the zeros, or places on the graph of the original function where the slope of the tangent line is zero. Terry constructed his graph of the derivative by first identifying points on the graph of the function where the slope of the tangent line is zero and then mapped those points on the graph of the derivative.
Terry further described the importance of the tangent line when graphing derivative functions:

T: In terms of how the tangent is, that's all that goes through my head, is where the tangent would lie on the function, whether it's positive or not; that puts the f prime graph, then, either above or below the x axis, and that's pretty much how I think about it, in terms of graphing it.
I: O.K. So when you look at this, the first thing that you think or the thing that you're thinking about is tangent lines.
T: Yeah.
I: So, what, specifically, about tangent lines are you thinking about?
T: Slope, the slope of the tangents.
I: And that will inform you to how to draw the derivative graph?
T: A basic outline, yeah.
I: All right. What else would come into play in drawing a derivative graph, or is there anything else that comes into play?
T: The intervals, too, that it's increasing on, you've got to worry about that. Because, like, the extrema points tell you where you have to worry about it going from a positive to a negative, so that's where, like I said, that is where it crosses the x axis. And those are really the two things that I think about that.
Terry demonstrated this process in his solution to Derivative Task 1:

T: The first things I look for on this, are the extreme points, because the graph of f prime, the extreme on the function's graph is going to be zero on the graph of f, and it looks like it's two and a half blocks out, so it's going to be zero there.

I: So, how do you know that at the extreme points your derivative graph is going to be zero?

T: Because the tangent on the graph, the slope of that is zero. So it has to cross the x axis on the f prime graph. And the first interval, from the y axis to the first extrema point, the derivative is positive, so it's going to look something like this. Then it goes negative, to that point, something like—and this is a minimum there. And then after that it starts to increase again. Something in the neighborhood of that.

Terry asserted that he had trouble working with and understanding concavity.

I hate second derivatives, only because they always say you can find out intervals of concavity, and I don't get that. I have to go from the first derivative and do things, always, in terms of that, in increasing and decreasing. I don't get concavity at all. I can see it after the fact, but as for drawing it out, it doesn't help at all.

Terry indicated that because he approaches the graphical problems working with tangent lines, the notion of concavity doesn’t fit in with his approach and it therefore troublesome.

It doesn't--because it seems like it's doing two steps inside of one, instead of--it would seem easier to visualize concavity, just because you go, okay, it's concave down, so it's got to be negative, the second derivative does. But I don't picture it that way. I guess I do --everything is in terms of the tangent line, so I do the second derivative in terms of the tangent line to the first derivative.

Terry was not as comfortable working with the notion of concavity because he could not fit it into his concept image of derivative as the slope of the tangent line.

Terry also worked comfortably with integrals in a graphic representation. Terry easily computed areas under curves using estimation procedures and Riemann sums. On
an examination question that presented velocity and time data for a car and asked questions about the distance the car traveled, Terry wrote that “The distance the car traveled is equal to the area under the velocity curve.”

Terry easily sketched graphs of the anti-derivatives of functions given a graph of the original function. Terry appeared to have a good grasp of the relationship between a function and its derivative. For example, consider the following passage in which Terry begins solving Derivative Task 3:

This is increasing—for the first interval, it's got to be increasing, because the \( g' \) is positive, so the slope of the tangent on this function is going to be positive. And it looks like it's getting—the slope's getting smaller, so it's probably like this. It crosses—it goes through zero...

Notice that Terry stated that the function must increase where the derivative is positive. Terry’s ability to appropriately discuss the relationship between a function and its derivative indicates that his concept image of derivative includes a rich understanding of derivative and anti-derivative.

Terry competently used physical properties and conditions of problems to help him solve them. For example, Terry was able to graph the velocity function from the position function and the acceleration function from the velocity function on homework assignments, examinations, and in-class activities. Terry correctly solved the following problem on the first examination, pictured in Figure 17. Notice that Terry did not mention the velocity versus time plot in his explanation for his acceleration versus time plot. He merely states that, “A negative acceleration fits the model.” Terry also used the derivative to find the position and velocity of an object at a certain time, given a formula for the acceleration of the object.
A student walks beside a 2-meter measuring stick, beginning her walk at the origin. Then she moves with decreasing speed toward the 2 meter mark. After coming momentarily to rest near the 2 meter mark, the student immediately begins moving toward the 0 meter mark with increasing speed. For each of the plots below, sketch graphs of this motion and briefly explain why you drew the plots as you did.

![Figure 17: Terry's Solution to a Kinematics Examination Problem](image)

The symbolic representation also contributed to Terry’s concept images of derivative and integral. Consider Terry’s concept map of derivative and integral (See Appendix E). Terry’s concept map of derivative and integral included many examples of rules for differentiation and integration. Furthermore, Terry answered most symbolic
derivative and integral problems correctly on examinations, homework assignments, and in-class activities. Terry competently applied the power rule, product rule, quotient rule, and chain rule to compute derivatives of polynomials, exponential and logarithmic functions, and trigonometric functions. The symbolic integral tasks involved the substitution method of solution with polynomial functions, exponential and logarithmic functions, trigonometric functions, and inverse trigonometric functions.

Terry competently solved derivative and integral problems in numeric representations on examinations, homework assignments, and in-class activities. On the following problem that appeared on an examination, Terry correctly estimated the derivative of \( f(1) \) from a table of average rate of change values for the function \( f(t) \) over different time intervals.

\[
\text{For } f(t) \text{ the sequence of values of } h \text{ approaching zero and the corresponding values of the average rate of change from } t = 1 \text{ to } t = 1 + h \text{ are given in the following table.}
\]

<table>
<thead>
<tr>
<th>( h )</th>
<th>Average Rate of Change of ( f(t) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/5</td>
<td>9.4932</td>
</tr>
<tr>
<td>1/25</td>
<td>8.3110</td>
</tr>
<tr>
<td>1/125</td>
<td>8.0992</td>
</tr>
<tr>
<td>1/625</td>
<td>8.0576</td>
</tr>
<tr>
<td>1/3125</td>
<td>8.0493</td>
</tr>
</tbody>
</table>

Working with the same function, \( f(t) \) presented above, Terry correctly determined that the function was increasing at \( t = 1 \). Terry wrote that, “The slopes of secant lines are positive as \( h \) approaches 0. This means the tangent lines to that point will be positive, therefore it is increasing.” Terry’s ability to work with derivative and integral in the numeric
representation indicates that his concept image of derivative and integral also included numeric presentations.

Finally, Terry's conceptualization of the Fundamental Theorem of Calculus was a link between his understanding of derivative and integral. Terry connected the concepts of derivative and integral in his concept map of derivative and integral with a symbolic statement of the Fundamental Theorem of Calculus. Furthermore, when I asked Terry during the third interview what 'integral' meant to him, he responded in the following way: "Anti-derivative. The reverse of derivatives. When you take a derivative, you should be able to go backwards. A reversing process." Terry's response to my question asking him to define 'integral' is an example of how Terry connects the concept of integral to the derivative.

Derivative and Integral Concept Image: Summary. Terry worked comfortably with derivative problems in a variety of representations. Terry's concept images of derivative and integral appeared to be balanced; that is, Terry worked proficiently with the symbolic, graphic, numeric, and physical representations of derivative and integral. Terry did not exhibit a preference for working with one representation over others, but his concept map of derivative and integral included many references to the symbolic representation.

Summary

Terry was classified as a Language-Mixer in the categories of Average Rate of Change and Derivative and Integral. Terry's concept images of average rate of change, derivative, and integral were balanced; that is Terry showed evidence of his competence working with these calculus concepts in a variety of representations. Terry's concept
map of rate of change included references to the everyday language used to describe rates of change, thus corroborating evidence that Terry used both mathematical and physical language to describe his conceptualization of average rate of change. Terry’s concept map of derivative and integral included a symbolic statement of the Fundamental Theorem of Calculus, evidence that Terry’s conceptualizations of derivative and integral are linked by this important theorem.

**Todd**

**Background**

Todd is a physics major. He entered the University of New Hampshire as an undeclared College of Engineering and Physical Sciences major and indicated early in the Fall semester that he was interested in majoring in Mechanical Engineering or Physics. Todd chose to major in physics because he enjoys physics. He indicated that he plans to work in industry and then eventually teach. Todd has a part-time job working as a research assistant for the nuclear physics group at the University of New Hampshire. He reported that he works approximately eight hours a week. Todd indicated that he will continue to work for the physics department over the summer and hopes to travel to Los Alamos next year with the nuclear physics group.

During the Fall semester, Todd enrolled in the Calculus/Physics course, an Honors seminar on the history of complex numbers and *Introduction to the College of Engineering and Physical Science*, a course required of all first-year, undeclared College of Engineering and Physical Science majors. In the Spring semester, Todd enrolled in
Introductory English, Introductory Linguistics, and Introductory Cultural Anthropology, in addition to the Calculus/Physics course.

Todd was home-schooled all of his life. Todd studied derivatives and antiderivatives in the calculus portion of his curriculum for a year. Prior to his study of calculus, Todd studied algebra, geometry, and trigonometry. Todd mentioned that he did not care for proofs. Todd studied physics for two years. The physics that he studied in 1998 included topics ranging from Newtonian physics to Einstein's Theory of Relativity. Todd specifically mentioned his study of springs during this time. Todd also studied physics in 1999, a year before he entered his first year in college. During his second year of studying physics, Todd used the text *The Mechanical Universe* (Frautschi, Goodstein, & Apostol, 1986).

Todd reported that he really enjoyed physics and particularly liked working with more theoretical ideas such as relativity. Todd also reported that he was very interested in geometry and numbers and mentioned that he enjoyed reading Gleick's *Chaos: Making a New Science* (1988).

Todd indicated that he decided to enroll in the calculus/physics program after a dean in the College of Engineering and Physical Sciences encouraged him to do so. He credited the atmosphere of the Calculus/Physics class with helping him adjust after being homeschooled. Todd is involved with the Juggling Club and Fencing Club on campus. In his free time, Todd enjoys juggling, reading, and hanging out with his friends. Todd was very interested and enthusiastic about participating in the clinical interviews.
Todd received a B in the fall semester of calculus. Twenty five out of 48 students in the Calculus/Physics class received a grade in the range of B- to B+. Todd received a B+ in his first semester of physics.

Physics Use Classification

Overview. Todd was classified as an Example-User in the category of Average Rate of Change and a Non-User in the category of Derivative and Integral. Todd talked about examples of physics problems or concepts as he worked through average rate of change problems and often used examples to justify his answers to average rate of change problems. Additionally, Todd’s concept map of rate of change included examples from kinematics.

Todd did not appear to use physics in a concrete way to help him conceptualize calculus concepts. Unlike his work with average rate of change, Todd did not use physics examples to help him understand or justify his answers to derivative and integral problems. Rather, Todd’s methods of solution and language usage as he solved derivative and integral problems were strictly mathematical.

The next two sections present evidence for Todd’s classifications as an Example-User in the category of Average Rate of Change and as a Non-User in the category of Derivative and Integral.

Average Rate of Change: Example-User. The physical representation played an important, but not dominant role in Todd’s concept image of Average Rate of Change. Todd referenced physical examples as he solved many average rate of change problems. For instance, Todd frequently used physical examples to clarify his answers to average
rate of change problems. Consider Todd’s explanation of his solution to Average Rate of Change Problem 5:

Well, for me that 3 means that it’s increased its speed or position — whatever this graph is — by 3. So maybe it’s moved 3 spaces forward on a checkerboard or maybe it’s a car going 3 miles per hour faster. It’s just something that’s increased by 3.

Todd’s answer to the above problem became meaningful to him as he considered a physical situation, namely increasing speed or position. Todd used the example of a checkerboard and a car to clarify the meaning of his answer. In a later problem, when I asked Todd to explain why an average rate of change of zero made sense as an answer he again evoked physics examples to explain his reasoning. Consider Todd’s justification of Average Rate of Change Problem 7,

Well, the average rate of change is... just... the total of that... it’s like if we go forward 3, 3 feet and then walk back 3 feet, we’ve really exerted a lot of effort but haven’t gotten anywhere. We’ve... had an average movement of zero. So move forward positive 3 and then we’d move negative 3. It’d total out to zero. This is the same thing. We’ve gone down all the way to zero from 9 and then back up. But we really haven’t moved anywhere.

Todd used the example of moving away from and back to a certain location to make sense of traveling zero distance in this problem. Notice that Todd introduces the language ‘average movement’ to justify the zero displacement. Todd used physics situations to justify his solution to this problem.

Todd’s concept map is also indicative of his use of physics examples to conceptualize average rate of change. (See Appendix D for a copy of Todd’s Rate of Change concept map.) Todd specified physics as one of the principal ways he thinks about rate of change. Furthermore, Todd included the relationship between position, velocity, and acceleration on his concept map of rate of change. Todd often used the
concepts of position, velocity, and acceleration to justify his solutions to calculus problems.

**Derivative and Integral: Non-User.** Todd did not appear to rely on his knowledge of physics concept or examples to help him conceptualize the concepts of derivative and integral. Furthermore, Todd tended to use strictly mathematical language as he solved both derivative and integral problems. During the second interview, I asked Todd if he ever thought about physics concepts as he worked through problems asking him to sketch derivative and anti-derivatives. Todd replied:

> It's just basically, if I'm looking at just a simple graph, I look at it as mathematical functions, you know, slope line and all this. But if it's like if it's velocity and actual problem-solving problem; all right, we have a car here speeding along until it hits a tree, and the problem asks me to plot the velocity. Say okay, you do that. That's when I think about different velocities and stuff. It's the context of the situation. I mean, it doesn't help me to think of these [Derivative Tasks 1-4] in physical ways. This way it's just, I just have to remember the tangents and the slopes, this is how it relates to this point and all that. In the other one it's, okay, acceleration goes like this; so the velocity is going to be increasing. Use the same principles and the same techniques; just think about it in a slightly different way, even though it's the same problem.

Todd indicated that the context of the problem directly affected how he approached a problem. Todd's statement that he approached graphical derivative and integral problems in one way and physical derivative and integral problems in another way is consistent with his solutions to interview tasks as well as in-class, examination, and homework problems. Todd seemed to work though graphical derivative and integral problems using properties of graphs, physical derivative and integral problems using properties of physics, and symbolic derivative and integral problems using formulas. In a forthcoming section, I further discuss how Todd solved problems in various representations.
**Summary.** Todd primarily used physics concepts and experiences as examples to justify his mathematical calculations of average rate of change. The physical representation played an important, although not dominant, role in Todd’s conceptualization of average rate of change. Todd’s concept map of rate of change points to the place of the physical representation in his conceptualization of average rate of change. Todd seemed comfortable working with the physical, symbolic, and graphic representations of average rate of change.

Unlike his conceptualization of average rate of change, Todd typically did not use physics to help him make sense of the derivative and integral concepts. Todd claimed that the context of a problem influenced how he conceptualized the problem. Thus, Todd seemed to use physics to help him solve derivative and integral problems if they were embedded in a physical representation. In the next section, I discuss my interpretation of Todd’s concept images of average rate of change, derivative, and integral.

**Concept Image**

**Overview.** In this section, I will discuss Todd’s concept images of average rate of change, derivative, and integral. I attempted to re-construct Todd’s concept images by using his concept maps as well as his responses to interview tasks, homework assignments, examinations, and in-class activities.

**Average Rate of Change.** Todd primarily used the physical and graphical representations to conceptualize average rate of change. As previously discussed, Todd often used physics examples and experiences to help shape his conceptualization of average rate of change. In particular, Todd closely connected the idea of time with rate of change. For instance, on the Average Rate of Change Pretest, Todd wrote that “Rate
of change describes how much things change in how much time.” He also wrote that when he hears ‘rate of change’ he thinks of acceleration.

Todd often used physics concepts to talk about the properties of rate of change. In his concept map of rate of change, Todd broke the central concept, rate of change, into three components: physics, graphical, and analytical. He indicated that the physical aspects of rate of change, which he wrote as position, velocity, and acceleration, are related to each other via derivatives and anti-derivatives. In addition to using physics examples to help conceptualize average rate of change, Todd also evoked graphical examples as he spoke about average rate of change. Todd seemed very comfortable working with the graphical representation of rate of change, and appeared to prefer working with this representation as much as or more than the other representations. For instance, during a discussion about the expression \( \frac{f(x_1) - f(x_0)}{x_1 - x_0} \), Todd drew a graph to help explain his thinking. A copy of Todd’s drawing is shown in Figure 18.

![Figure 18: Todd’s Graphical Depiction of Average Rate of Change](image)
The passage below is an excerpt of Todd’s description of his graph and how it relates to the above expression.

OK, I’ve just graphed out the, the axes, the vertical one is f(x) and the horizontal x. Draw an approximate function, which I’ll label f(x). I’m going to put in on the x-axis a little starting point, which will be x₀. And I’m going to have a finishing point which is x₁. I’m going to take those points and plot them up to my function and these point will respectively be (x₀, f(x₀)) and the second one will be (x₁, f(x₁)) . All right. And drawing the lines down from the points to...where they are on the graph. They’re on the horizontal line from the lower point over to the...other line. And this space right here is the...f(x₁) minus f(x₀) , which is merely the change in the outputs. But this, the f(x₁) minus the f(x₀) over x₁ minus x₀ is, for all intents and purposes, the change in y over the change in x; the slope between the two points. That’s how I see it.

Todd easily explained how he graphically envisioned the concept of average rate of change. Likewise, Todd appropriately mapped symbols (such as x₁, \( \frac{\Delta y}{\Delta x} \)) to their graphical counterparts on his sketch. This is consistent with Todd’s concept map of Rate of Change in which he broke down the central concept of rate of change into physical, graphical, and analytical. Not only does Todd seem to think about these three representations as ways to conceptualize rate of change, but he also acknowledges connections between the representations.

Further evidence of the presence of strong connections between representations in Todd’s conceptualization of average rate of change came when Todd discussed why he thought average velocity was a rate of change. In the next passage, Todd evoked physical, graphic, and symbolic representations to explain his thinking about why average velocity is a rate of change.

Well, the way I look at it is that velocity is the rate of change of the position. It’s like you move 3 meters in a second after 1 second, you move 3 up on the graph. And average velocity, since velocity is a rate of
change, is an average rate of change. So it’s the average rate of change equation – the delta a over delta b.

Notice that Todd talks about physical examples (moving 3 meters in a second), graphical situations (‘you move 3 up on the graph’), and symbolic representations (‘it’s the average rate of change equation – delta a over delta b’). Another example of Todd’s ability to connect the physical, graphic, and symbolic representations is Todd’s solution to a homework problem pictured in Figure 19. Todd used his knowledge that average velocity is equal to change in distance over change in time to solve the above problem. Todd used the physical context of the problem to justify his answers with units. Furthermore, Todd realized that the situation presented in Part (c) was not physically possible, another instance of Todd’s use of real-world examples and experiences to helping him solve average rate of change problems.

Todd seemed comfortable working with average rate of change strictly in a symbolic context. Todd correctly computed the average rate of change between \( t = 1 \) and \( t = 1 + h \) of \( f(t) = t \), \( g(t) = t^2 \), and \( h(t) = t + t^2 \) on his in-class calculus activities. Todd also correctly computed the average rate of change of \( f(t) = 2t^2 - t \) from \( t = 1 \) to \( t = 3 \) on his final examination.

Todd seemed to have some difficulty working with the numeric representation of average rate of change. Todd frequently converted tabular data into graphs in order to solve numeric average rate of change problems. For example, Todd sketched out a graph in order to help him answer Average Rate of Change Problems 9.1 and 9.2. The data for Average Rate of Change Problems 9.1 and 9.2 was presented in a table. Todd converted the data in the table to a graph before he answered problems 9.1 and 9.2.
15. A car travels for 30 miles with an average velocity of 40mph and then for another 30 miles with an average velocity of 60mph. [6, p. 149]

(a) What is the average velocity of the car for the entire trip?
\[ \frac{\Delta d}{\Delta t} = \frac{60 \text{ mi}}{1.5 \text{ h}} \]
\[ \Delta t = 1.5 \text{ h} \]
\[ \Delta x = 30 \text{ mi} + 30 \text{ mi} = 60 \text{ mi} \]
\[ \bar{v} = 48 \text{ mph} \]

(b) Another car travels for 30 minutes at 40mph and then for 30 minutes at 60mph. Find the average velocity over the 1-hour time period.
\[ \frac{\Delta d}{\Delta t} = \frac{50 \text{ mi}}{1 \text{ hour}} \]
\[ \bar{v} = 50 \text{ mph} \]
\[ \frac{40 \text{ mi}}{1 \text{ h}} \cdot \frac{1 \text{ h}}{0.5 \text{ h}} = 80 \text{ mi} \]
\[ \frac{60 \text{ mi}}{1 \text{ h}} \cdot \frac{1 \text{ h}}{0.5 \text{ h}} = 120 \text{ mi} \]

(c) A car is to travel 2 miles. It went the first mile at an average velocity of 30mph. The driver wishes to average 60mph for the entire 2-mile trip. Is this possible? Explain.

\[ \frac{1}{2} \text{ hour} = \frac{2}{30 \text{ mph}} + \frac{1}{V} \]
\[ \frac{1}{V} = \frac{2}{60 \text{ mph}} - \frac{1}{60 \text{ mph}} = 0 \]
\[ \frac{1}{V} \neq 0 \]

Figure 19: Todd's Solution to an Average Rate of Change Homework Problem

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Todd also converted tabular data into graphs on some of his in-class activities and homework. Todd also exhibited some difficulty working with data in a tabular form on tests. For instance, on the following examination question, Todd attempted to use the formula for average rate of change to solve the problem.

For \( f(t) \) the sequence of values of \( h \) approaching zero and the corresponding values of the average rate of change from \( t = 1 \) to \( t = 1 + h \) are given in the following table.

<table>
<thead>
<tr>
<th>( h )</th>
<th>Average Rate of Change of ( f(t) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/5</td>
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</tr>
<tr>
<td>1/125</td>
<td>8.0992</td>
</tr>
<tr>
<td>1/625</td>
<td>8.0576</td>
</tr>
<tr>
<td>1/3125</td>
<td>8.0493</td>
</tr>
</tbody>
</table>

Find the average rate of change from \( t = 1 \) to \( t = 1 + 1/25 \) and explain its meaning.

Todd used the formula Average Rate of Change = \( \frac{\Delta x}{\Delta t} \) to try to solve the problem. Todd wrote that \( \frac{\Delta x}{\Delta t} = \frac{8.045 - 8.3119}{1/2} \). He did not realize that the average rate of change from \( t = 1 \) to \( t = 1 + 1/25 \) was simply the value given in the ‘Average Rate of Change of \( f(t) \)’ column adjacent to 1/25.

Average Rate of Change: Summary

Todd tended to use physics examples to justify answers to average rate of change problems or help him make sense of average rate of change problems. Todd frequently
justified his answers to problems during the interviews without prompting to do so. Todd used physical situations and examples to justify the reasonableness of his answers. Todd seems to make connections between the physical, graphical, and symbolic contexts of average rate of change. In his concept map of rate of change, he explicitly connected the three components with joining lines.

Although Todd worked well with most average rate of change representations, he had some difficulty working with the numeric representation of average rate of change. Todd tended to avoid working with the numeric representation of average rate of change problems. He typically converted numeric data into graphs in order to solve such problems. Because of Todd's avoidance of the numeric representation, his concept image of rate of change seems to be unbalanced; that is, Todd possesses strong understandings of the physical, graphic, and symbolic representations of average rate of change. However, he seems to hold a weak conceptualization of the numeric representation of average rate of change.

**Derivative and Integral.** Similar to his work with average rate of change, Todd seemed to exhibit a preference for working with derivatives and integrals in a graphical context. On a number of occasions he mentioned that the derivative is the slope of the tangent line, and quite literally solved graphical derivative and anti-derivative problems this way. Consider this passage from Todd's solution to Derivative Task 1:

So the derivative is basically the slope of a tangent line at a point. So I'm going to take a few points, plot the, where the tangent line, the numerical value is, and then do a basic graph from there. So first off I'm going to find the points where slope equals zero. That is this high point here, which I will call A. And the other point is B down on the bottom. And I'm going to plot it at about maybe two and a half on the second graph. And -- And let's see. Slope going to be negative at this point. And that's also where it's going to change directions. This will be a minimum on this
I'm going to approximate the slope of it by doing rise over run. So it's about two, three, four, five, six up and three overs. So that's going to be a negative three. So I'll plot negative three on the graph at somewhere close to that point. And I'm going to have to try to find a point -- for A, and it looks to be a high slope. Just about one half on the graph. It's at four, or thereabouts...closer to five. And I'm going to plot that. Put that there. And now I'm going to do another one, switching point, which I didn't see because it crosses the x axis again on the first try. Increasing, so it's changing direction. It's going to be about a slope of one. It's going to be somewhere around there. Should be about up to there on the graph.

Todd’s strategy for graphing the derivative from a graph of a function was first to locate the points on the graph of the function where the slope of the tangent line is zero and mark those points on the graph of the derivative. Then he chose other points on the graph of the function and estimated the slopes of the lines tangent to the curve at those points. He then plotted the tangent line slope estimates on the graph of the derivative. He continued this process until he had a general idea of the shape of the derivative graph.

The markings on Todd’s solution to the examination question pictured in Figure 20 also indicate that Todd regularly uses this strategy to produce the graph of a derivative given a graph of a function. Notice the string of dots on Todd’s acceleration graph. Todd first drew in the dots and then connected them with the acceleration curve. The notion of derivative as slope of the tangent line seemed to dominate Todd’s conceptualization of derivative.

Todd also competently and comfortably used graphs to represent his conceptualization of the integral. During Todd’s third interview, as he worked on Integral Tasks 2 and 3, Todd sketched out his graphical interpretations of the two problems. Copies of Todd’s work are presented in Figures 21 and 22. Figure 21 is a copy of Todd’s graphical interpretation of Integral Task 3. Todd shaded the area under
6. (15 both pts) Sketch functions for the acceleration and position given the velocity below. Take $x(0) = 6$ m.

![Graph](image-url)

Figure 20: Todd's Solution to a Graphical Derivative Problem
the graph of \( y = x^2 \) to represent the solution to the definite integral \( \int_{0}^{5} 3x^2 \, dx \). The graph in Figure 22 is Todd’s graphical interpretation of \( \int 3x^2 \, dx \). Todd made a connection between his sketch in Figure 22 and his sketch in Figure 21 by saying, “The area, as \( x \) increases, is going to follow the function \( x^3 \). So the area, or graph of the area, is just basically the indefinite integral.”

![Figure 21: Todd’s Graphical Interpretation of \( \int_{0}^{5} 3x^2 \, dx \)](image)

When I asked Todd to describe the difference between Integral Tasks 2 and 3, he stated that the former gives a function and the latter gives the area under the curve.
The way I, I see this \[ \int 3x^2 \, dx \] is this, well, to use book terms, this is an indefinite integral. This is just a shape, general graph of it. You’re given initial values or something to tell where it is on the graph... And the second one \[ \int_0^5 3x^2 \, dx \] is a definite integral, and it’s asking you to find...OK, this is the integral from here to here...and it’s kind of asking you to find the value of maybe the original function, or the, the area under the original function. That’s how I see it....You can see it and solve it, it’s the area under that graph.

Todd easily worked through Integral Task 2, which asked him to find the anti-derivative of \( 3x^2 \). He indicated that the answer would be \( x^3 + C \) and joked that the \( + C \) was necessary otherwise points would be taken off. He claimed that “We know the shape of it, but we don’t exactly know the shape of it.” Todd indicated with an up-and-down motion with his hands that the constant would move the graph vertically in the plane.

Figure 22: Todd’s graphical interpretation of \( \int 3x^2 \, dx \)
Although Todd did not seem to depend on physics concepts or examples to help him conceptualize derivative and integral problems, he used physical contexts and parameters if they were included in the problem statement. For example, Todd effectively used his knowledge of calculus and physics to solve an examination problem shown in Figure 23. Notice the consistency of Todd's explanations with respect to the position, velocity, and acceleration graphs he sketched. Todd used the physical situation to appropriately justify his graphs. For instance, Todd claimed that, "The students velocity is always decreasing..." in order to justify his velocity versus time graph.

Todd appeared comfortable computing most derivatives and integrals using formulas. Todd exhibited some difficulty computing derivatives and integrals of logarithmic and exponential functions on his examinations. For example, on a number of different occasions, Todd was unable to compute the derivative of functions of the form \( y(t) = c^t \), where \( c \) represents a constant. On his final examination, Todd computed

\[
\int \frac{1}{\sqrt{1-x^2}} \, dx = \ln(\sqrt{1-x^2}) + C.
\]

On the same examination, Todd also computed

\[
\int \cos(t) \, dt = \frac{1}{|t| \sqrt{1-t^2}}.
\]

Todd was fairly consistent in his ability to compute derivatives on examinations and homework assignments. Occasionally, Todd made careless mistakes when computing derivatives of complicated functions or when using the chain rule.

Todd correctly solved most derivative and integral problems presented to him in a numeric context on homework assignments, examinations, and in-class activities. For example, on examinations, homework assignments, and in-class activities, Todd computed lower and upper estimates for the distance an object traveled from a velocity.

A student walks beside a 2-meter measuring stick, beginning her walk at the origin. Then she moves with decreasing speed toward the 2 meter mark. After coming momentarily to rest near the 2 meter mark, the student immediately begins moving toward the 0 meter mark with increasing speed. For each of the plots below, sketch graphs of this motion and briefly explain why you drew the plots as you did.

**Position vs. time**

![Position vs. time graph](image)

**Brief Explanation:**
- The student's position increases until she reaches the 2 meter mark, then decreases back to 0.

**Velocity vs. time**

![Velocity vs. time graph](image)

**Brief Explanation:**
- The student's velocity is always decreasing with respect to the origin.

**Acceleration vs. time**

![Acceleration vs. time graph](image)

**Brief Explanation:**
- The student's change in velocity, or acceleration, is a constant, negative.

Figure 23: Todd's Solution to a Kinematics Examination Problem

versus time table. Additionally, Todd was able to calculate the derivative of a composition of functions from function data presented in a table. Todd did not appear to
convert tabular data into a graph or other representation, as he did for numeric average rate of change problems. Rather, Todd seemed comfortable working with data directly from a table.

Finally, Todd rarely mentioned the Fundamental Theorem of Calculus during any of the interviews or in his responses to homework problems, in-class activities, or examination problems. In fact, even though Todd drew his concept maps of derivative and integral together, it is not the Fundamental Theorem of Calculus that connects the concepts of derivative and integral; rather, Todd connected the two concepts with the trigonometric functions. Todd did not talk about the integral as an inverse derivative operation as some other students in the present study talked about the integral.

**Derivative and Integral: Summary.** Todd proficiently worked with various representations of the derivative and integral. Todd did not regularly use physics to help him solve derivative and integral problems. Rather, he strictly used mathematics ideas and terminology when discussing derivative and integral problems during the interviews. Todd seemed to have graphical visualizations at hand for many problems, even if he did not always call upon those visualizations to help him solve a problem. Todd’s concept images of derivative and integral seemed to be balanced; that is, Todd’s conceptualization of derivative and integral seemed to include various representations. However, The Fundamental Theorem of Calculus did not seem to play a role in Todd’s conception of derivative and integral.

**Summary**

Todd was classified as an Example-User in the category of Average Rate of Change and as a Non-User in the category of Derivative and Integral. Todd appeared to
have an unbalanced concept image of average rate of change. Todd seemed to avoid using the numeric representation when solving average rate of change problems. Todd’s concept map of rate of change included the physical, graphic, and symbolic representations, but excluded the numeric representation, further evidence of Todd’s unbalanced concept image of average rate of change.

Todd’s concept images of derivative and integral were balanced; that is Todd showed evidence of his competence working with these calculus concepts in a variety of representations. Todd did not use physics to help him conceptualize the derivative and integral concepts. Rather, Todd appeared comfortable using the mathematical definitions of these concepts to solve problems and discuss his solutions.

Travis

Background

Travis is a Mechanical Engineering major who chose Mechanical Engineering as his major because he was interested in designing golf equipment. Golfing is one of Travis’ hobbies and a high school teacher encouraged him to pursue his interests related to golfing and golf equipment. Travis indicated that he is becoming more interested in managerial aspects of engineering rather than designing golf equipment.

Travis reported that he decided to enroll in the Calculus/Physic class because the small learning environment appealed to him. He stated that he feels he learns better in smaller classes. Travis enrolled in Engineering Design and Graphics and an Honors seminar focusing on the history of complex numbers during the Fall semester, in addition
to the Calculus/Physics course. In the Spring semester, Travis enrolled in a Mechanical Engineering class and *Principles of Microeconomic*, in addition to the Calculus/Physics course.

Travis indicated that he was not involved in any clubs, sports (not even golf), or organizations on campus and that he did not have a job during the semester. His hobbies include running and playing the guitar. He hopes to obtain a summer internship, possibly focusing on mechanical engineering.

Travis enrolled in a year-long Advanced Placement calculus class during his senior year in high school. He reported that his class spent a good deal of time reviewing precalculus topics but also covered derivatives and integrals. He indicated that he thought his teacher did not convey concepts to the student very well and that he thought the pace of the class was too slow.

He also enrolled in a year-long physics class during his senior year in high school. Travis enthusiastically talked about projects that the class worked which involved the construction and building of objects. Three projects that Travis specifically talked about were building a pasta crane, building a machine, and working with electronics kits.

Travis received a B+ in the first semester of calculus. Twenty five out of 48 students in the Calculus/Physics class received a grade in the range of B- to B+. Travis received a B in his first semester of physics.

**Physics Use Classification**

**Overview.** Travis was classified as a Language-Mixer in the categories of Average Rate of Change and an Example-User in the category of Derivative and Integral. Travis was classified as a Language-Mixer in the Average Rate of Change category.
because he frequently interspersed physics terminology with mathematical vocabulary as he worked through calculus problems. Travis was classified as an Example-User in the Derivative and Integral category. Travis talked about examples of physics problems or concepts as he worked through derivative and integral problems and often used examples to justify his answers to derivative and integral problems.

The next two sections present evidence for Travis’s classification as a Language-Mixer in the Average Rate of Change category and an Example-User in the Derivative and Integral category.

**Average Rate of Change: Language-Mixer**

Travis frequently used calculus and physics vocabulary as he talked about his solutions to calculus problems. For instance, when I asked Travis to talk about the expressions \( \frac{f(x_i) - f(x_0)}{x_i - x_0} \) and \( f(x_i) - f(x_0) \), he blended language from both calculus and physics. Speaking about the former expression, Travis said, "I solved this sort of expression in both calculus and physics and I think it means rate of change." Travis talked about a change in time during his discussion of the latter expression: "I think it’s just a change problem...which I’ve seen in both calculus and physics. It’s a change not taking into account a change in time." Furthermore, Travis’ answers to the Average Rate of Change Problems 5 – 7 reflect his connection between time and rate of change. For each of the questions 5 – 7, Travis calculated the correct answer and then wrote ‘units/unit time’ after each answer.

Travis reported that when he heard the word ‘rate’, he thought of the amount of things done per unit of time. For example, Travis stated, “Like, if I eat three apples per hour then the rate is the amount of things done per, in that case, hour.” On his Average
Rate of Change Pretest, Travis indicated that he used the concept of rate of change in both high school calculus and physics and that he thought about rate of change as “final minus initial per unit time”. When I asked Travis if he worked with rate of change more in calculus or physics in high school, he said:

I think I’ve worked with it more in physics class, because that’s where we talked about velocity and acceleration and average acceleration and average velocity and change in position and stuff like that. So we did, like, we did a lot of that ‘cause it, that’s what physics is. So I worked with it more in...Well, I worked with it in calculus, too, but I...sometimes I didn’t know I was working with it. With derivatives and stuff I wasn’t really sure what was going on that whole time! (both laugh) So I might have worked with it and then...(both laughing). What can I say?

Notice that the rate of change concept is more meaningful to Travis when conceptualized as velocity or acceleration. Travis’s admission that he didn’t always know that he was working with rate of change in his high school calculus class is evidence that the physics interpretations of rate of change were more meaningful to him. Travis not only indicated that his understanding of rate of change was grounded in physics, but also suggested that what he learned in calculus class was disconnected from his everyday experiences. A more detailed discussion of Travis’s conception of average rate of change will be presented in a forthcoming section.

**Derivative and Integral: Example-User.** Travis tended to use physical examples and situations to help him understand and interpret derivative and integral problems. Travis often made statements during his interviews that led me to believe he sometimes thought about physical examples to interpret derivative and integral problems. For example, after solving Derivative Task 2, Travis said, “If I can’t think of like a physical situation that it [the graph] would correspond to, it’s hard to picture what’s going on.” Later, when I presented him with Derivative Task 4, Travis recalled a similar problem on
a Calculus/Physics examination he took two weeks earlier. Travis remarked, “That’s just like a problem we had on the test. We had to do position, acceleration, and velocity. It took me like an hour!” Travis recalled his work on a problem involving kinematics as he solved this abstract mathematics problem. Travis recognized that he was using the same fundamental ideas to solve the mathematical problem at hand as well as the physics problem he previously solved on the examination.

Travis also used physics concepts and examples to help him make sense of the concepts of derivative and integral. During the third interview, I asked Travis what he thought about when he heard the word ‘integral’. Travis responded, “It makes me think a lot of going from velocity to position because we did a lot of that in physics.”

Although Travis made statements about physical examples helping him understand derivative and integral problem situations, Travis sometimes became confused trying to remember the relationship between position and velocity. During his second interview, Travis mentioned that the relationship between position and velocity “is confusing”. However, later in the year, while he worked on Final Interview Task 2, Travis remarked on the relationship between position, velocity, and acceleration: “The velocity is going to be the derivative of that [position], the acceleration is going to be the derivative of whatever the velocity is.” In a forthcoming section, I discuss how Travis solved derivative and integral problems in various representations.

**Summary.** Travis was classified as a Language-Mixer in the average rate of change category since he frequently blended calculus and physics vocabulary when expressing his conceptualization of average rate of change. Travis’s understanding of
average rate of change seemed to be grounded in his physical experiences with average rate of change.

Travis was classified as an Example-User in the Derivative and Integral category. Travis frequently used examples to help him understand calculus problems or justify his work. As he solved calculus problems, Travis sometimes recalled specific physics problems that shared the same underlying concepts as the calculus problem. He then would draw upon his understanding of physics to help him solve the calculus problem.

In the next section, I discuss my interpretation of Travis's concept images of average rate of change, derivative, and integral.

Concept Image

Overview. In this section, I will discuss Travis's concept images of average rate of change, derivative, and integral. I attempted to re-construct Travis's concept images by using his concept maps as well as his responses to interview tasks, homework assignments, examinations, and in-class activities.

Average Rate of Change. As previously described, Travis frequently used physics terminology in his discussions of calculus problems. Although Travis used both physics and mathematical language when discussing calculus problems, he also appeared comfortable using various representations of average rate of change to solve problems. Travis appeared most comfortable working with the graphic and symbolic representations of average rate of change. On his concept map of rate of change, Travis constructed two main branches describing average rate of change: Analytical and Graphical. (See Appendix D for a copy of Travis's Rate of Change concept map.) Travis's preference for using the symbolic and graphic representations when solving problems was also
manifested in his interviews, classwork, homework, and examinations. Travis seemed confident in his understanding that the average rate of change is the slope of the secant line. He often mentioned slope of the secant line when talking about average rate of change problems. For example, as he discussed his solution to Average Rate of Change Problem 5, Travis said:

So, um, to find the average rate of change between \( x = 0 \) and \( x = 2 \), you just connect the points with the secant line at, um, 1, it’s going to be \((1, 1)\) and \((2, 4)\). And you take the slope of that which is the rise over the run. So it’s, the rise is 3 and the run is 1, so it’s 3 units, is the average change on that one.

Notice that Travis began his solution by connecting the points on the graph with the secant line and then calculating the slope of the secant line. Travis also interspersed the phrases ‘average rate of change’ and ‘slope of the secant line’ as he solved Average Rate of Change Problem 6.

Find the average rate of change between negative 1 and 2. You do the same thing, you connect \((-1, 1)\) with \((2, 4)\). And you take the slope of the, the line that connects those, which is... the rise is 3 and the run is 3. So that’s one unit, is the average rage of change.

Finally, Travis used features of the graph to help him solve Average Rate of Change Problems 5 – 8. Consider Travis’s solution to Average Rate of Change Problem 7:

On number 7, um, since this is, this... graph has symmetry about the y-axis, between negative 3 and 3 there’s going to be zero ... the average rate of change is going to be zero because there’s a horizontal line between those two points. The slope of that is zero.

Notice that Travis used the symmetry of the graph to recognize that the y-coordinates of the points in question were the same. Travis also mentioned a horizontal line connected
the two points and used the fact that the slope of a horizontal line is zero to solve the problem.

Travis’s concept map of rate of change also shows his understanding of the connection between average rate of change and the slope of the secant line. On his Rate of Change concept map, Travis stated that average rate of change can be thought of graphically as the ‘secant line between two points’. Travis also included a pictorial representation of the secant line on his concept map.

The other main branch describing Average Rate of Change on Travis’s concept map was ‘analytical’ rate of change. Travis indicated that the average rate of change can be thought of analytically as the “average value between two points on [an] interval” and computed using the formula \( \frac{y_2 - y_1}{x_2 - x_1} \). Travis seemed very comfortable using a formula to calculate average rate of change. Travis was the only student to recognize that his answer to Average Rate of Change Problem 8 was equivalent to the expression discussed in Average Rate of Change Problem 2.

And to find the average rate of change between \( x = a \) and \( x = b \), you would use the, um, rate of change function, I think, whatever it’s called: \( \frac{f(a) - f(b)}{a - b} \). Which is what was illustrated on the page before that, it’s just to find the average rate of change of a function.

While Travis proficiently solved problems in graphical and symbolic representation, he did not exhibit a conceptualization of the connections between the graphical and symbolic representations. For instance, in the above passage, Travis referred to the
expression \( \frac{f(a) - f(b)}{a - b} \) as the ‘rate of change function’ yet he did not attempt to reconcile the function \( \frac{f(a) - f(b)}{a - b} \) with his understanding of slope as rise over run.

Travis competently solved average rate of change problems presented to him in a numeric representation on the Average Rate of Change Pretest and on his examinations. For example, on the following examination problem, Travis correctly computed the average rate of change to be 8.3119.

For \( f(t) \) the sequence of values of \( h \) approaching zero and the corresponding values of the average rate of change from \( t = 1 \) to \( t = 1 + h \) are given in the following table.

<table>
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</tr>
<tr>
<td>( 1/3125 )</td>
<td>8.0493</td>
</tr>
</tbody>
</table>

Find the average rate of change from \( t = 1 \) to \( t = 1 + 1/25 \) and explain its meaning.

Travis also correctly identified that the function, \( f(t) \) was increasing at \( t = 1 \). He stated, "As \( [h] \) gets smaller, the average rate of change gets smaller. So, the further out you go, the greater the slope, thus it is increasing."

**Average Rate of Change Concept Image: Summary.** Travis used physics language in his solutions to many average rate of change problems. As Travis solved average rate of change problems during the interviews, he often incorporated both mathematics and physics terminology into his discussions of the problems. Travis did
not necessarily submerge the average rate of change problems in a physical context, rather he used the physics language to help him describe the problem situation and his solution.

Travis proficiently solved average rate of change problems in physical, graphic, symbolic, and numeric representations. Travis seemed to prefer working with the graphic and symbolic representations of average rate of change. On his Rate of Change concept map, Travis focused on the graphical and algebraic representations of rate of change.

**Derivative and Integral.** Travis exhibited a balanced concept image of derivative and integral. Travis appeared comfortable and confident when solving derivative and integral problems in a variety of representations. As previously discussed, Travis frequently used physics examples to help him make sense of certain derivative and integral problems and his answers to those problems. Although physics examples played an important role in Travis's understanding of derivative and integral, his conceptions of derivative and integral were not dominated by the physical representation.

Travis seemed very comfortable working with the graphic representation of derivative and integral. For example, Travis mentioned a graphical interpretation of derivative in his concept map of derivative. (See Appendix E for a copy of Travis's Derivative Concept Map.) He stated that the derivative "describes rate of change" and is "equivalent to the instantaneous slope of a function". During the interviews, Travis claimed that "A derivative of a function is a graph of its slope at every point."

Travis also included a graphic representation of integral on his integral concept map. He stated that the integral is "equivalent to area under a curve" and sketched an example function. Travis used the graphic representation of integral to understand his
answers to integral tasks. For instance, after Travis solved Integral Task 3, I asked him to explain what he answer meant. He said, "That would be the area of the function."

Like many of the other students, Travis seemed to follow a procedure when graphing derivative and anti-derivative functions. He began by locating where the slope of the original function changed from positive to negative and related that to a change in the derivative graph from positive function values to negative function values. Travis described his method for sketching the derivative function during his solution to Derivative Task 1:

I found out where the sloping changed from being positive to negative. And that's a derivative function, goes from being positive to negative. And where it crosses the x, where the original function crosses the x-axis, that means something. I'm not sure what it means yet. I forgot.

Travis indicated that the extreme points of a function are significant because they correspond to zeros on the graph of the derivative of the function. The significance Travis places on zeros and extreme points of functions was also evident in his work on the in-class activities, homework, and examinations. For example, in the examination question pictured in Figure 24, Travis marked the maximum of the velocity graph, the roots of the acceleration graph, and the minimums of the position graph — evidence of the importance of these point to him as he sketched out the derivative and anti-derivative.
6. (15 both pts) Sketch functions for the acceleration and position given the velocity below. Take \( x(0) = 6 \) m.

Figure 24: Travis's Solution to an Examination Problem
When problems were posed to Travis in a physical context, he seemed to rely on his understanding of physics to solve the problems. For example, when I presented Travis with Final Interview Task 2, his first reaction was to sketch a diagram of the spring. See Figure 25 for a copy of Travis's work on Final Interview Task 2. Travis began to solve the problem by talking about what the shape of the position graph would look like. Travis said, "It's going to behave sinusoidally. And it's going to be damped by um, an exponential function."

2. If a spring is hanging vertically from a pole with a 20kg weight on the end and you pull slightly on the weight and then let go, what would the position, velocity, and acceleration plots of the weight look like?

Figure 25: Travis's Work on Final Interview Task 2

Travis initially attempted to derive a formula for the position so that he could use differentiation rules to derive the formulas for velocity and acceleration. When he had difficulty attempting to derive a formula for position, I prompted him to think about the shape of the position graph:

I: Do you have any idea what the graph of the position might look like for this problem?
T: Um, it should...it’s just going to be a wave, damped...until it damps out to zero.
I: If that’s generally what the position looks like, do you have an idea of what the velocity and acceleration might look like?
T: Um...(long pause). I think the, um, I know the velocity and acceleration are going to be opposite each other. It’s [the spring] moving downwards and it’s going to be accelerating upwards. I don’t know. I think it’s going to be some weird function. Well it’s going to be, whatever it is it’s going to be, the velocity is going to be the derivative of that [position], the acceleration is going to be the derivative of whatever the velocity is. I’m not really sure what that would be right now....the acceleration would resemble the position.

Notice that Travis talked about the physical properties of the spring; when the spring moves downward, the acceleration will be upward. Using his knowledge of the physical system and the relationship between position, velocity, and acceleration, Travis was able to sketch out graphs of the position, velocity, and acceleration of the spring. However, Travis ignored his statements that the position and acceleration should be opposite one another. His graphs of position and acceleration are identical.

Travis was quite successful at computing derivatives and integrals using rules and formulas. Travis competently used the Power Rule, Product Rule, Quotient Rule, and Chain Rule to compute derivatives of polynomial functions, logarithmic and exponential functions, and trigonometric functions. Travis successfully computed the integrals of polynomial functions, exponential and logarithmic functions, and trigonometric functions using the substitution method of integration. Travis confidently solved Integral Task 2, asking him to compute the anti-derivative of \(3x^2\). Travis explained that, “You add one to the power and divide by that new number so the threes cancel and you just get one.” Travis later adds a ‘ + C’ to his answer on Integral Task 2 and explains that, “It’s just a constant. You can shift the graph up or down.” When I ask him why the ‘ + C’ was
necessary, Travis claimed: “Because you lose the constant term when you take a derivative...you lose the constant term. So when you integrate, you have to give the constant back.” Travis’s remarks about the constant of integration indicate that he possesses an understanding of both the graphical and symbolic representations of integral. However, Travis seemed to prefer working with derivatives in a symbolic context. During the second interview, Travis said that he found working with derivative problems in a symbolic context “a lot easier” than working with derivatives in a graphical context. Travis stated, “I’m not a very visual person.” Furthermore, although Travis mentioned the graphic representation on his derivative and integral concept maps, he devoted a major portion of his concept maps to the rules for differentiation and integration. On his Derivative concept map, Travis lists the Power, Chain, Product, and Quotient rules and states the general form of each rule. On his Integral concept map, Travis gives examples of how to integrate polynomial, logarithmic, and trigonometric functions. Travis’s concept map, along with his work on interview tasks and other class activities indicates that he is most comfortable working with the symbolic representation of derivative and integral.

Travis demonstrated proficiency solving derivative and integral problems in a numeric representation on examinations, homework assignments, and in-class activities. For example, Travis correctly estimated the derivative of f(t) from a table of average rate of change values for the function f(t) over different time intervals and he computed lower and upper estimates for the distance an object traveled from a velocity versus time table. Travis did not appear to use graphs to help him interpret data presented to him in tables; rather he worked with the data directly from the tables.
Finally, Travis rarely mentioned the Fundamental Theorem of Calculus during any of the interviews or in his responses to homework problems, in-class activities, or examination problems. Travis drew separate concept maps for derivative and integral and did not mention the Fundamental Theorem of Calculus on either of his concept maps. Travis also did not talk about the integral or anti-derivative as an inverse derivative operation as some other students in the present study talked about the integral. The Fundamental Theorem of Calculus seemed to play an insignificant role in Travis’s understanding of derivative and integral.

**Derivative and Integral Concept Image: Summary.** Travis’s concept images of derivative and integral appeared to be balanced; that is, Travis appropriately solved derivative and integral problems in various representations. Travis used physics concepts and situations as examples to help him make sense of some derivative and integral problems. Although Travis used physics example to help him make sense of derivative and integral problems, he seemed to have a preference for working with the symbolic representation. Travis’s concept maps of derivative and integral, along with statements he made during the interviews and his in-class work verify Travis’s preference for working with the symbolic representation of derivative and integral.

**Summary**

Travis was classified as a Language-Mixer in the Average Rate of Change category and an Example-User in the Derivative and Integral category. Travis blended mathematics and physics vocabulary as he talked through his solutions to average rate of change problems. Travis did not tend to submerge derivative and integral problems in a physical context in order to solve them, rather he uses examples from physics to help him
understand certain aspects of calculus concepts. Travis’s concept images of average rate of change, derivative, and integral appeared to be balanced.

Michelle

Background

Michelle is a mathematics major who describes herself as very driven. Michelle indicated that she sets high standards for herself and enjoys school. She stated that she plans to continue her education and get a Masters and Ph.D. degree, “even if my job doesn’t require it.”

Michelle decided to major in mathematics because she likes mathematics and really enjoyed her high school Advanced Placement calculus class. She was always enrolled in honors mathematics classes in high school and talked about being ahead of her peers in high school mathematics classes. Michelle hopes to obtain a summer actuary internship and in upcoming summers would like to become involved in a mathematics research internship.

Michelle decided to enroll in the Calculus/Physics class because she wanted to avoid large lecture-style classes. She reported that she was excited to hear in the recruitment letter for the class that the class was more involved than the other calculus and physics classes. That is, Michelle perceived that the Calculus/Physics course would push her to think about calculus in new ways, especially in relation to physics. Michelle indicated that the Calculus/Physics classes seemed like a “step up” from the regular
classes and she was looking for a more fast-paced class. She was eager to enroll in a class where she would not be “doing the same stuff [from high school] over again.”

Michelle reported that she was involved in Pi Mu Epsilon, the undergraduate mathematics honors club, during the fall semester, but she was less involved during the spring semester because of academic commitments. In addition to Calculus/Physics, Michelle enrolled in *Introduction to Cultural and Social Anthropology* and a course on the history of the great psychologists during the fall semester and *Introduction to Scientific Programming* and *Introductory English* during the spring semester.

Michelle indicated that her busy academic schedule did not leave her much free time during the week, but on the weekends she liked to go out or watch TV with her friends. During the summer and on school breaks, Michelle works at a yogurt shop in her hometown.

Michelle is very energetic and eager to participate in the interview sessions. Michelle was not shy about asking for advice on classes to take and often asked about my own experiences as a mathematics major. I was impressed by her maturity in thinking about steps to take as an undergraduate to insure her success after college. Even as a first-year university student, she was certain about her plans to continue on to graduate school.

Michelle enrolled in year-long Advanced Placement calculus class during her senior year in high school. She reported that the class covered limits, derivative, anti-derivatives, implicit differentiation, and related rates problems. She indicated that she especially enjoyed working on optimization problems in calculus class. Michelle also
enrolled in a year-long physics class in high school. She mentioned on a number of occasions that she didn’t particularly like physics.

Michelle received an A in the fall semester of calculus. Thirteen out of 48 students in the Calculus/Physics class received a grade in the range of A- to A. Michelle received an A- in her first semester of physics.

Physics Use Classification

Overview. Michelle was classified as a Language-Mixer in the Average Rate of Change category and a Non-User in the Derivative and Integral category. Michelle blended mathematics and physics vocabulary as she talked through her solutions to average rate of change problems. Michelle did not appear to use physics in a concrete way to help her conceptualize derivative and integral problems. Michelle’s methods of solution and language as she solved derivative and integral problems were strictly mathematical.

The next two sections present evidence for Michelle’s classification as a Language-Mixer in the Average Rate of Change category and a Non-User in the Derivative and Integral category.

Average Rate of Change: Language-Mixer. Michelle’s responses to many of my questions during the first interview were peppered with physics terminology. For instance, consider Michelle’s response when I asked her to talk about what the expression $f(x_1) - f(x_0)$ meant to her. Michelle answered: “Just, that would be change in position, right? Or... that’s how I would interpret it. Or change in... some position of two different functions... or wait. No! Two different points. Same function, but two different points.” Michelle interspersed ‘change in position’ with the mathematical terminology ‘function’.
Michelle also associated time with the notion of rate of change. Michelle indicated that when she hears the word ‘rate’ she thinks about “the time it takes someone or something to do something.” She indicated in numerous places that the notions of rate of change and time are naturally connected. However, Michelle made statements that she did not see either the notion of time or rate of change having to do with physics in a meaningful way to her. When we talked about her experiences using rate of change in her high school classes, Michelle indicated that she worked with rate of change in both calculus and physics, but that the physics content coverage was less meaningful to her.

I understood it and I was, I enjoyed it more in the calculus ‘cause that was the stuff I like with the boxes, the rate of change [related rates problems] and, but physics...I know we definitely talked about it in physics, but it didn’t really mean much to me, basically.

Michelle indicated that the physics she learned was not very meaningful to her outside of the physics classroom. However, the language Michelle used as she worked on problems and some notions prevalent in her concept map show otherwise. For example, Michelle indicated on her rate of change concept map that “constant velocity means a zero acceleration” and “area under the curve for velocity functions gives the distance.” These examples show that Michelle may be making more meaningful connections between calculus and physics than she would like to admit.

Derivative and Integral: Non-User. Michelle did not rely on her knowledge of physics to help her solve derivative and integral problems. Michelle tended to use strictly mathematical language as she solved both derivative and integral problems. During the second interview, I asked Michelle to explain the relationship between a function and its derivative. Michelle replied, “The derivative is the slope. For instance [consider] a linear function, so that means the slope is going to be like a y = a line. It’s one less
exponential power. I don’t know how to say it!” Later, when I asked Michelle if she ever thought about graphical derivative and anti-derivative problems in terms of physics concepts or examples such as position, velocity, and acceleration, she claimed:

Not really. I don’t think, like, velocity, acceleration, position. But I know usually that’s how it is. This would be position, that would be velocity, that would be acceleration. But I don’t think in terms of that, usually, unless it’s like physics or specifically asks for the acceleration or velocity.

Michelle indicated that the context of the problem directly affected how she approached a problem. Michelle’s statement that she approached velocity and acceleration problems thinking about the concepts physically is an indication that the problem representation plays an important role in how Michelle approaches problems. Michelle seemed to work through graphical derivative and integral problems using properties of graphs, physical derivative and integral problems using properties of physics, and symbolic derivative and integral problems using formulas. In the forthcoming sections, I discuss how Michelle solved derivative and integral problems in graphic, physical, symbolic, and numeric contexts.

Summary. Michelle interspersed physics and mathematics vocabulary as she solved average rate of change problems. Although Michelle used physics language as she discussed her solutions to average rate of change problems, she often talked about the how she did not readily connect physics concepts and experiences to her conceptualization of average rate of change.

Michelle did not use physics to help her make sense of the derivative and integral concepts. Michelle claimed that the context of a problem influenced how she conceptualized the problem. Thus, Michelle only used physics to help her solve derivative and integral problems if they were embedded in a physical representation.
In the next section, I discuss my interpretation of Michelle’s concept images of average rate of change, derivative, and integral.

**Concept Image**

**Overview.** In this section, I will discuss Michelle’s concept images of average rate of change, derivative, and integral. I attempted to re-construct Michelle’s concept images by using her concept maps as well as her responses to interview tasks, homework assignments, examinations, and in-class activities.

**Average Rate of Change.** Michelle appeared to have an unbalanced concept image of average rate of change early in the Fall 2000 semester. In particular, Michelle exhibited a weak conceptualization of the graphic representation of average rate of change. For example, during the first interview, Michelle never talked about the relationship between the slope of the secant line and the average rate of change. At first, Michelle would not attempt to answer Average Rate of Change Problems 5 – 8. She claimed that since she didn’t know the formula to solve them, she couldn’t attempt to answer the questions. After Michelle worked through Average Rate of Change Problems 9.1 and 9.2, I prompted her to return to questions 5 – 8 and try to answer them. Her first reaction to my suggestion was:

Why? Is it the same graph, change in position over change in time to figure out this stuff? Cause this isn’t, this isn’t necessarily a velocity graph. I guess [my work on problems 9.1 and 9.2] does [relate back to problems 5 – 8], but I wouldn’t know which equation to use or how to go about it if I didn’t know...what it was. For this 9.1 and 9.2], the only way I did this was knowing it was position, a position and time graph. So I think it would be easier if I knew what...Just like if it was a velocity-time graph, I would be figuring out the acceleration by doing the change, so...
When I prompted Michelle to think about if she could choose to label the axes what she wanted in order to solve the problem, she gave a nice explanation about why she could put her own context on the problem:

I: OK. Is it possible when we do problems, if we’re given nothing with the axes, if we’re just given some graph, are we allowed to put labels on the graphs? Can we think of things in that way or will that change the problem?
M: It might not, actually. To do it, to do this one just assuming it’s position?
I: Right, yeah, yeah. Will that change the problem at all?
M: Um...no. I don’t think so.
I: OK. Why not?
M: (both laugh) Because it...it doesn’t matter. Oh, wait, yeah, actually, cause it doesn’t matter what graph it is. Like, for this one, it’s position.
It’s... the, um...you’re finding out the velocity by doing the change of this [position] and the change of this [time]. So regardless what, you don’t know what your finding, which exact unit you’re finding, but it, it would be the change of one axis toward the change of the other.

Michelle was able to generalize her work with average velocity to solve Average Rate of Change Problems 5 - 8. She very nicely described the generalization as “the change in one axis toward the change of the other.”

Later in the semester, Michelle appeared to have built a connection between average rate of change and the slope of the secant line. Michelle connected the notion of slope to the central Rate of Change concept on her concept map of Rate of Change. Furthermore, Michelle used the relationship between average rate of change and the slope of the secant line to answer a number of problems on her in-class activities.

Michelle seemed comfortable solving average rate of change problems presented to her in physical contexts. That is, Michelle confidently solved average rate of change problems that included physical events or constraints in the problem statements. Recall that during the first interview, Michelle was unable to solve Average Rate of Change
Problems 5 – 8, problems which asked her to find the average rate of change between different sets of points on the graph of a parabola. However, Michelle quickly and correctly solved Average Rate of Change Problems 9.1 and 9.2, dealing with average velocity. Michelle explained her solution to Average Rate of Change Problem 9.1:

I know average velocity is change in position over change in time, right? So from 0 to .2, which is right here, it would be change in position, which would be position 2 minus position 1, which is .5. So it would be .5 over change in time, it’s, um, point or zero to .2, it’s .2 seconds.

However, sometimes Michelle ignored her intuition about how to solve average velocity problems and instead attempted to rely on mathematical rules from memory. When I presented Michelle with Average Rate of Change Problem 10, she was unable to solve the problem, even though she just successfully solved Average Rate of Change Problems 9.1 and 9.2. She said:

Um...the velocity would be the slope, the derivative of this. But, um...Ah. I know...I learned this last year. I know to sketch the derivative it's, I think it's above and below [the x-axis] of the derivative equals, wait. Above and below [the x-axis] of the derivative equals increasing and decreasing of the function. Or it's the other way around. I can't remember...I know it's some rule like that, but I just can't remember it.

Notice that Michelle attempted to rely on her memory of taking derivatives in her high school calculus class. Michelle seemed more comfortable trying to recall her prior methods of solution than attempting to apply her knowledge of average rate of change.

Michelle comfortably solved average rate of change problems in a numeric representation. Michelle was able to calculate the average rate of change or average velocity of an object from data presented in a position versus time table on in-class activities, interview tasks, and homework problems. Furthermore, Michelle correctly answered the following examination question:
For \( f(t) \) the sequence of values of \( h \) approaching zero and the corresponding values of the average rate of change from \( t = 1 \) to \( t = 1 + h \) are given in the following table.

<table>
<thead>
<tr>
<th>( h )</th>
<th>Average Rate of Change of ( f(t) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/5</td>
<td>9.4932</td>
</tr>
<tr>
<td>1/25</td>
<td>8.3110</td>
</tr>
<tr>
<td>1/125</td>
<td>8.0992</td>
</tr>
<tr>
<td>1/625</td>
<td>8.0576</td>
</tr>
<tr>
<td>1/3125</td>
<td>8.0493</td>
</tr>
</tbody>
</table>

Find the average rate of change from \( t = 1 \) to \( t = 1 + 1/25 \) and explain its meaning.

Michelle interpreted her answer, 8.3119, as “the slope of \( f(t) \) from \( t = 1 \) to \( t = 1 + 1/25 \)”.

Average Rate of Change Concept Image: Summary. Early in the semester, Michelle exhibited an unbalanced concept image of average rate of change; that is, the graphic representation of average rate of change did not seem to be very meaningful to Michelle. However, Michelle’s performance on examination problems, homework assignments, and in-class activities indicate that she developed an understanding of average rate of change that included the graphic representation. Although Michelle did not seem to make a connection between average rate of change and the slope of the secant line during her first interview, she used the slope of the secant line to solve average rate of change problems on homework assignments, examinations, and in-class activities. By the end of the Fall 2000 semester, Michelle competently solved most average rate of change problems presented in graphic, physical, symbolic, and numeric representations.
Derivative and Integral. As previously described, Michelle did not rely on physics concepts to help her conceptualize the derivative and integral. Michelle competently solved derivative and integral problems in multiple representations. Michelle was especially comfortable working with derivatives and integrals in a graphical context. She followed a procedure when graphing derivatives or antiderivatives from the graph of a function. Michelle described this procedure as:

Well, I do-the two things, the thing that I keep on saying, the above and below of the derivative means increasing and decreasing of the function. And then, also, the derivative is the slope.

Michelle seemed to use the fact that the derivative is the slope of the tangent line to identify places on the graph of the function where the slope of the tangent line was zero. For example, she wrote in the words “horizontal tangent” on her work on the examination question shown in Figure 26. Then she mapped those points to zero on the horizontal axis of the derivative graph.

What is most striking about Michelle’s language as she solved graphic derivative problems is the repetition of the phrase “above and below of a derivative is increasing and decreasing of a function.” For example, consider a part of Michelle’s discussion of her solution to Derivative Task 2. Michelle’s graph is shown in Figure 27.

So then, from this point, from B to C [cusp to discontinuity], that’s increasing, so that’s going to be above. ... Let’s see. And then, this one, it’s increasing from, say, D to E [discontinuity to maximum], which means it’s going to be above. And then, from -- it’s actually decreasing until right here [end of graph], which is F, so this is going to be below, on the derivative, so it’s probably just going to be above, so it’s D, and then from E to F it’s below.

Even on her concept map of rate of change, Michelle included a derivative portion and a spoke off of the derivative concept is “Above and below the x-axis of the derivative =
6. (15 both pts) Sketch functions for the acceleration and position given the velocity below. Take \( x(0) = 6 \text{ m} \).

Figure 26: Michelle's Solution to an Examination Problem

increasing and decreasing of its anti-derivative.” (See Appendix E for a copy of Michelle's Derivative and Integral Concept Map.) Another spoke from the main concept
of derivative in her concept map is that of "tangent line is slope". Michelle also drew a
graphic example to show what she meant by the tangent line. Michelle’s concept map of
Derivative and Integral corroborate evidence from her interviews, examinations,
homework assignments, and in-class activities that points to her rich conceptualization of
the graphic representation of derivative and integral.

Figure 27: Michelle’s Solution to Derivative Task 2
Michelle also seemed to understand the notion of integral as area under a curve. Consider Michelle’s work on the in-class activity picture in Figure 28. In Part (a), Michelle easily concluded that Car 2 traveled a greater distance because it “has greater distance under the curve.” Michelle correctly solved similar problems on examinations and homework assignments.

3. Two cars start from rest at a traffic light and accelerate for several seconds. The following graph shows their velocities vs. time. [3, p. 308]

![Graph showing velocity vs. time for two cars](image)

(a) Which car is ahead after one second? How do you know?

Car 2 has greater distance under curve.

(b) Which car is ahead after two seconds? How do you know?

Car 1

Figure 28: Michelle’s Solution to a Calculus Activity Dealing with Integrals

During the third interview, Michelle talked about indefinite integrals giving the area under a curve. She said, “If you want to find the area of the distance, which is, when...
you take the anti-derivative of a function that give you the area under the curve.” I asked Michelle to clarify what she meant by ‘finding the area of the distance’. She responded by saying, “Well, it [anti-derivative] gives you the area under the curve. And if it’s a velocity function, then it gives you the distance.” However, later when I asked her why we would want to find the area under the curve, Michelle responded by saying, “I don’t know actually.” Michelle seemed to exhibit an understanding of what the answer to an integration problem meant, but she did not have an understanding of the motivation for computing an integral.

Michelle appeared very comfortable using the symbolic representation of derivative and integral to solve problems. As she worked through graph problems, she often attempted to identify the functions with a symbolic expression. For example, as she worked on Derivative Task 1, Michelle stated, “And also what you know is that this is--what do they call it, cubic function, right? So that means that this derivative is going to be a polynomial.” I believe that Michelle meant to say ‘quadratic’ or ‘parabola’ when she said ‘polynomial’.

Michelle confidently computed derivatives using the Power Rule, Product Rule, Quotient Rule, and Chain Rule. On her examinations, homework assignments, and in-class activities, Michelle calculated the derivatives of polynomial functions, logarithmic and exponential functions, and trigonometric functions using the derivative formulas. Additionally, Michelle included the general form of the power rule in her concept map of Derivative and Integral. Michelle also mentioned various methods and rules of integration in her concept map. In one circle, Michelle wrote that ‘\[ \text{Integral} = \text{add one to the exponent and then divide by the new exponent} \]’. The majority of the concepts

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related to integral on Michelle’s concept map deal with rules for integrating functions.

Michelle computed anti-derivatives of polynomial functions, logarithmic and exponential functions, and trigonometric functions using substitution on homework assignments, examinations, and in-class activities. She even used the derivative to check her answers.

Consider the following passage from Michelle’s third interview in which she was solving the problem $\int 3x^2 \, dx$.

The anti-derivative of $3x^2$, what the rule is to increase the exponent by 1, so that would be $x^3$, and then you divide the function or whatever, you divide it by the new exponent, so that way you’d be dividing by 3, so it’s $3$ divided by 3 really equals the 1, so it’s $x^3$. And then you can check it by taking the derivative of it by multiplying it by the exponent, so it would be $3x$, and then you subtract 1 from the exponent, so it would be 2, and that matches.

This passage is representative of Michelle’s ability and eagerness to solve symbolic derivative and integral tasks.

Michelle did not seem as comfortable working with derivatives and integrals in a physical context as compared to other representations. Consider Michelle’s solutions to the examination question pictured in Figure 29. In her answer to this problem, Michelle seemed to ignore the physical phenomena and obtained graphs of the position, velocity, and acceleration functions based on her knowledge of calculus. Her position versus time function is a cubic and she seemed to sketch out her velocity versus time and acceleration versus time graphs by using her knowledge that the velocity is the derivative of the cubic position function, and the acceleration is the derivative of the quadratic velocity function. It appeared that she attempted to explain her graphs based on her understanding of the calculus. In this problem, Michelle’s confusion of velocity with speed was evident. Additionally, she seemed to confuse acceleration and velocity. Michelle answered

A student walks beside a 2-meter measuring stick, beginning her walk at the origin. Then she moves with decreasing speed toward the 2 meter mark. After coming momentarily to rest near the 2 meter mark, the student immediately begins moving toward the 0 meter mark with increasing speed. For each of the plots below, sketch graphs of this motion and briefly explain why you drew the plots as you did.

**Position vs. time**

**Velocity vs. time**

**Acceleration vs. time**

**Brief Explanation:**

The position is always increasing with respect to time.

The first motion, the velocity is decreasing, and for the second part of the trip, the velocity is increasing.

The first way from 0m to 2m, the student is walking with a decreasing speed which means the acceleration is negative. After when the student is walking back from 2m to 0m, the speed is increasing so the acceleration is positive.

Figure 29: Michelle's Solution to a Kinematics Examination Problem
another examination question that asked when an object in one dimension must be slowing down as when “the acceleration is negative”. Michelle’s physics misconceptions seem to interfere with her ability to solve some derivative problems in a physical context.

Michelle also had some difficulty converting physical entities into mathematical symbols and formulas. Toward the end of the first semester, Michelle and her two partners were working through an activity during class. The activity was designed to walk the students through finding the center of mass of a 2 meter-long rod using Riemann sums and integrals. The students were told that the density of the rod was \( \lambda(x) = \frac{1}{2} x + 1 \) kg/m. The students were also given the following formula for finding the center of mass:

\[
x_{cm} = \frac{\sum m_i x_i}{\sum m_i},
\]

where \( m_i \) is a mass, and \( x_i \) is the center of mass of \( m_i \). The students had calculated the total mass of the rod to be 3kg and had found the first moment of the rod. I approached Michelle’s group as they were attempting to find an expression for the exact center of mass of the rod. Michelle and her other partners asked me “how to do” the question asking them to find an expression for the center of mass of the rod. The group initially wanted to write \( \int \frac{3x}{3} \) for the center of mass. They seemed unsure about how to deal with the \( x_i \)'s in the numerator of the center of mass formula. The group reasoned that the \( x_i \) represented length, but they wanted to replace the \( x_i \) with the total length of the rod. There was confusion among the group members about how to treat the lengths of the pieces knowing the length of the rod. In this case, the group members did not know how to interpret the \( x_i \) from the center of mass formula. At first, they believed that the \( x_i \) represented the total length of the rod. The group seemed to ignore the meaning of the
subscript on the variable $x$. After the Calculus instructor and I talked with the group more about the problem, they were able to solve it.

Michelle’s conception of integral was closely tied to the Fundamental Theorem of Calculus. When I asked Michelle what the word “integral” meant to her, she responded, “Anti-derivatives, basically. And if you take the derivative of the anti-derivative, then it's just the function itself that you're taking the anti-derivative of, like that thing—the Fundamental Theorem of Calculus.” Additionally, in her concept map of integral, she wrote “If take the derivative of the integral you are just left with the original function” as a spoke off of the concept of integral. Michelle decided to draw her concept maps of derivative and integral together, but she did not connect the two concepts with the Fundamental Theorem of Calculus. Rather, she wrote “related very closely” to connect the two concepts. She only included the Fundamental Theorem of Calculus on the integral side of the concept map.

**Derivative and Integral Concept Image: Summary.** Michelle did not regularly use physics to help her solve or discuss derivative and integral problems. Michelle sometimes had difficulty translating physical situations into mathematical symbols and formulas as evidenced by some of her work on the in-class activities. Michelle’s concept images of derivative and integral did not seem to include a rich conceptualization of the physical representation.

Michelle proficiently worked with derivatives and integrals in a variety of other representations. Michelle seemed to prefer working with derivatives and integrals in a symbolic context. Her concept map of derivative and integral was dominated by formulas and rules for differentiating and integrating functions.
Summary

Michelle was classified as a Language-Mixer in the Average Rate of Change category and a Non-User in the Derivative and Integral category. Michelle blended physics and mathematics vocabulary as she talked through her solutions to average rate of change problems. Michelle’s concept image of average rate of change did not include a rich conceptualization of the graphic representation early in the Fall 2000 semester.

Michelle was classified as a Non-User in the Derivative and Integral category because she did not depend on physics concepts or examples to help her conceptualize derivative and integral problems. Michelle’s concept images of derivative and integral were unbalanced. Michelle seemed to avoid using the physical representation when solving derivative and integral problems. Rather, Michelle appeared comfortable using the mathematical definitions of these concepts to solve problems and discuss her solutions.

Paul

Background

Paul entered the University of New Hampshire without declaring a major in the College of Engineering and Physical Sciences. Paul reported that he plans on declaring himself a mathematics major with a concentration in physics, because he finds mathematics intriguing and rewarding and also hopes to pursue a degree in astrophysics.

Paul reported that he decided to enroll in the Calculus/Physics class because it seemed like it would provide him with a “good challenge”. He also perceived the class as offering him the chance to gain a more thorough understanding of the course material.
than taking the classes separately. During the Fall semester, Paul was enrolled in
*General Chemistry* and an introductory honors English class along with
Calculus/Physics. Paul was enrolled in *Microeconomics, Introduction to Scientific
Programming*, and Calculus/Physics during the Spring semester.

Paul reported in March that he was not currently involved with any organizations
or clubs on campus. Paul enjoys hiking, rock climbing and running and he plays guitar in
his free time. Paul did not have a job during the academic year and did not speak of
summer employment. Paul indicated that he hopes to travel to the western United States
next summer in order to work in astrophysics or astronomy research.

Paul reported that he had taken an AP Calculus class in high school that was
taught by a first year teacher. The class covered topics such as integration, derivatives
including the definition of derivative, and Riemann Sums. Paul also took a year-long
physics course. In this course, the class covered kinematics, motion, lasers, magnets,
optics, waves, and perpetual motion. Paul reported that he liked “pretty much
everything” in calculus, but that he didn’t get along with his physics teacher. Paul
reported that he didn’t like the teaching style of his physics teacher.

Paul received an A in Calculus and a B+ in Physics. Twenty three out of 48
students received a grade in the range of B- to B+ in Physics, and thirteen out of 48
students received a grade of A or A- in Calculus.

**Physics Use Classification**

**Overview.** Paul was classified as a Language-Mixer in the Average Rate of
Change category and a Non-User in the Derivative and Integral category. Paul was
classified as a Language-Mixer in the Average Rate of Change category because he
blended mathematics and physics vocabulary as he talked though his solutions to average rate of change problems. Paul did not appear to use physics in a concrete way to help him conceptualize derivative and integral problems. Thus, Paul was classified as a Non-User in the Derivative and Integral category. Paul’s methods of solution and language as he solved derivative and integral problems were strictly mathematical.

The next two sections present evidence for Paul’s classification as a Language-Mixer in the Average Rate of Change category and a Non-User in the Derivative and Integral category.

**Average Rate of Change: Language-Mixer.** Paul answered the Rate of Change Pretest question, “When you hear the word ‘rate’ what do you think of?” as “Rate of change. Usually derivatives and instantaneous rate of change.” When I asked him to elaborate on his answer during the first interview, Paul re-stated his original response and also indicated that he understood rate of change as related to motion.

OK. Um, when I hear the word ‘rate’ what do I think of? Um, usually rate of change or derivatives and instantaneous rate of change... But, um, rate, it’s the, um... how something’s moving, I guess, is kind of more what I’m getting into. Um, like how fast or... how fast, how slow....

Notice Paul’s mention of rate as a description of how something is moving. Describing rate as so closely related to motion is a physical way to interpret rate and could indicate that Paul’s previous experiences with rate have made a lasting impression on him. It seems that “how fast something is moving” had relevance to Paul’s past experiences.

When I asked Paul to talk about his work with the expression \( \frac{f(x_1) - f(x_0)}{x_1 - x_0} \), he immediately began talking about his experiences using this expression in his high school
physics class. The examples that he mentioned of his use of the expression in calculus all related to speed.

I would say that we used it in physics, as well as the calculus class. Um, when dealing with, like, the speed of a car on a ramp or something...when we...using that, or maybe the...marble on a roller coaster type thing. Um, but, we also did deal with it in calculus class. Um, I wasn’t sure where we used it, but I’m sure it was, at one point we did, like, ah, a problem with speed, like a word problem, I’m trying to figure...I think one of them might have been like, um,...a cop car is taking radar on a car and at one point he’s going — or you have to prove that at one point he’s going faster than the average that he was going, so that the cop could nail him, I guess, or something. (both laugh). And, I think, that was in there somehow.

Notice that Paul offered three different average rate of change situations: (1) The speed of a car on a ramp, (2) A marble on a roller coaster, and (3) A speeding car problem.

Paul also talked about the expression \( \frac{f(x_1) - f(x_0)}{x_1 - x_0} \) as the average rate of change formula.

And the expression \( \frac{f(x_1) - f(x_0)}{x_1 - x_0} \), um, is the average rate of change with \( x_0 \) being the first time that you take, or your first reading, and \( x_1 \) is the next in the sequence of your measurements.

Notice Paul’s description of the variables \( x_0 \) and \( x_1 \) as points of time in a sequence of measurements. Paul seemed to be thinking about a physical application of the average rate of change formula as he described what the expression meant.

**Derivative and Integral: Non-User.** Paul did not appear to rely on his knowledge of physics to help him solve derivative and integral problems. During the second interview, I asked Paul if he ever thought about physics concepts as he solved graphical derivative and anti-derivative problems. He said that he never had, but then remarked:
But sometimes I can see a function, like maybe negative \( x \) squared or something. I think of it as...the trajectory of the baseball thrown or a basketball or something like that. Just something coming...going up and coming down. I really have never thought of it as velocity and acceleration and stuff like that.

Paul had a very strong image of the derivative as the slope of the tangent line at points of a function. He comfortably talked about the prominence of slope in the relationship between a function and its derivative.

Always I look at the derivative as the slope of the original function. So, um, if it’s whether it’s [the derivative] negative or positive then I can figure out what the original function is doing at the negative part — what it’s [the function] slope is doing when it’s [the derivative] negative or positive... if you have a really, really large positive slope then you should have a high value for the \( x \) component at the point on the derivative, on the graph of the derivative. And the slope of the original graph is your value for the derivative.

Paul’s conception of derivative as a slope of the tangent line was also evident in his concept map of Derivative (see Appendix E). Paul mentioned that the derivative was the slope of the tangent line and also talked about other graphical aspects of the derivative, such as concavity and maximums and minimums. Paul noted on his concept map of Integral that the integral is the area under the curve. Paul seems to be very comfortable working with derivative and integral problems in various representations. I will further discuss Paul’s concept images of derivative and integral in an upcoming section.

**Summary.** Paul interspersed physics and mathematics vocabulary as he solved average rate of change problems. Paul’s past experiences with average rate of change seemed to have an impact on his conceptualization of average rate of change.
Paul did not use physics to help make sense of the derivative and integral concepts. Rather, Paul had a strong conceptualization of derivative as the slope of the tangent line and integral as the area under the curve of a function.

In the next section, I discuss my interpretation of Paul’s concept images of average rate of change, derivative, and integral.

Concept Image

Overview. In this section, I will discuss Paul’s concept images of average rate of change, derivative, and integral. I attempted to re-construct Paul’s concept images by using his concept maps as well as his responses to interview tasks, homework assignments, examinations, and in-class activities.

Average Rate of Change. Paul displayed proficiency solved average rate of change problems using the graphic representation. For instance, Paul easily computed the average rate of change between various points on a graph of \( f(x) = x^2 \) and found the average rate of change between two points on a position versus time graph. On the in-class activities, Paul was able to answer questions concerning where the average rate of change of a graph was the greatest, smallest, and closest to zero.

During the first interview, Paul indicated that he did not always connect the slope of the secant line with the average rate of change formula. He seemed to believe that although the average rate of change formula and the slope of the secant line yielded the same result, they were distinct approaches. Paul seemed to see a relationship between slope of the secant line and average rate of change since he began to solve many of Average Rate of Change Problems 5 - 8 by using a slope approach. However, he specifically mentioned in a number of instances, that he was using the ‘slope formula’ or
the ‘average rate of change’ formula — leading me to believe that, although he saw some connections between the two, he was still viewing them as distinct entities.

Consider the following passage where Paul stated that he was using the slope formula to compute the average rate of change. He mentioned that his high school teacher taught him to look at rise over run and that was how he remembered how to compute the slope of a line. Paul was able to compute the slope of the line using rise over run to guide him.

P: OK. Um, for the average rate of change between x is 1 and x is 2, I kind of look at the graph initially and see where the point x equals 2 brings you... so you have the point (2, 4) here and then I look at x is 1 and I see it’s about (1, 1) for the coordinate. And then, um, I went about this by using the slope formula, um, so I guess what I did was form a line here... ’cause I deal with lines much better, I guess! And so, um, our teacher always taught it as rise over run. So it’s pretty much the same thing as... change in your y over change in x, but, it’s just (mumbled)...
I: And that’s the slope of that line that’s....
P: Yes, the straight line between the two points, ’cause it’s going to continue on from that point.
I: OK
P: So then it’s.... um, the slope, m, is \( y_2 - y_1 \) over \( x_2 - x_1 \) or — filling this in... y over 2 minus 1....(mumbling)...so you have a slope of... 3. I think. Does that make sense?
I: OK. Does that make sense to you?
P: Yeah, yeah.

Notice that Paul stated that he “deals with lines much better”. This is consistent with his concept map of Rate of Change. (See Appendix D for a copy of Paul’s Rate of Change Concept Map.) Paul only drew two spokes from the major concept of Average Rate of Change. One spoke was the description, “Secant line of a function that intersects the function at two points.” The other spoke was the formula \( \frac{f(b) - f(a)}{b - a} \).

Consider the next passage where Paul begins to speak of average rate of change and slope as distinct. Paul was solving Average Rate of Change Problem 7 and he was
confused by the slope of a horizontal line. His previous method of computing rise over run broke down because he didn’t know how to deal with a line having no rise.

Um, and the average rate of change between $-3$ and $3$, um, at first I think at this one, I looked at $3$ and then I looked at $-3$ and they both bring you to, um... $9$ – if we’re doing $f$ of $x$ equals $x$ squared, I think. So, then my first reaction was I wasn’t quite sure how the rate of change was on this one. I didn’t think about how to draw the line, at first. Um, just ‘cause... I guess I’m not as used to seeing a horizontal line on this type of thing. And then, I think what I did here was the actual definition of the average rate of change. I did, um, $f$ of $b$ minus $f$ of $a$ over $b$ minus $a$. And then our first point is going to be... $9$ minus $9$ over $3$ minus $-3$. (pause) I’m trying to go too fast for my own self. So $9$ minus $9$ over $3$... so we’ll call that $b$. ....(mumbling) So we get $0$ over $6$ and that’s $0$ so your average rate of change is just going to come out to be zero. And I had to kind of think about that, if the average rate... if the average rate of change could be zero – and I decided that it was pretty valid!

Notice Paul’s language as he talked about the strategy he used instead of computing rise over run: “...I think what I did here was the actual definition of rate of change.” This indicates that Paul had not made a connection between his ‘rise over run’ strategy and using the definition of rate of change. Recall that Paul identified the expression

$$\frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

as the average rate of change formula that he used mainly to compute speeds.

In the next passage, Paul elaborated on his motive for using the ‘actual definition’ of the average rate of change.

I: OK. Um, and you mentioned also on this part, in number 7, that you used the rate of change formula.
P: Um-hum.
I: How does that differ from the slope formula? Or why did you choose to use that over...
P: Um, basically my initial reaction to this one [number 6] was to draw the [secant] line because they’re, because they’re close together and it’s something that I’m more used to see than a line intersecting this way – horizontally.
I: OK
P: Also because, I think, since this one has slope of zero, I didn’t really think rise over run. And that’s usually what I tend to do with other things like that.

Finally, I asked Paul to estimate the average rate of change between two points on a position versus time plot and he specifically mentioned a distinction between the average rate of change formula and the slope of the secant line. “When I’m looking at this, the curve dips down to almost –3, and if I draw a line there. That wouldn’t be my first reaction of how to do it. I would do with the average rate of change formula.”

Later in the semester, Paul appeared to make the connection between average rate of change formula and the slope of the secant line. For instance, on his concept map (see Appendix D), Paul related the concepts ‘Secant line of a function that intersects function at 2 points’ and \( \frac{f(b) - f(a)}{b - a} \) to the concept ‘Average [rate of change]’. Additionally, Paul displayed his knowledge of connection between the average rate of change formula and the slope of the secant line on homework assignments and in-class activities.

Consider Paul’s solution to the homework assignment pictured in Figure 30. In Part (c ), Paul wrote that “This [slope formula] is the same formula used to find the average velocity of the object…”. Paul seemed to make the connection between slope of the secant line and average rate of change.

Paul identified problems asking for average velocity or average acceleration as average rate of change problems. As Paul was solving Average Rate of Change Problem 9.1, which asked him to compute average velocity over an interval, he described his solution process as follows: “The average velocity for the interval 0 ≤ t ≤ 0.2. Um, let’s see, for this one I….stuck with the average rate of change trend.” When I asked Paul
13. The position $p(t)$ of an object moving along a line is given by the graph below. [2, p. 25]

(a) Estimate the average velocity of the object between times $t = 1$ and $t = 3$.

$$\text{avg vel} = \frac{2 - 1}{3 - 1} = \frac{1}{2}$$

(b) Find the equation of the secant line of $p(t)$ between times $t = 1$ and $t = 3$, and sketch the graph of the secant line on the plot above.

$$y = mx + b$$

$$2 = 3(1) + b$$

$$y = \frac{1}{2}x + 3$$

(c) Write down the formula you used to find the slope of the secant line in part 13b. Compare the formula with the one you used to find the average velocity in part 13a.

$$\frac{2 - 1}{3 - 1} = \frac{\Delta y}{\Delta x}$$

This is the same formula used to find the average velocity of the object over distance.

(d) For what times, $t$, is the object's velocity positive? For what times is it negative?

The velocity of the object is positive from $t = 0.5$ to $2.5$, and negative from $0$ to $1$ and from $2.5$ to $3.5$.

Figure 30: Paul's Solution to an Average Rate of Change Homework Problem
how he knew that average velocity was an average rate of change, he responded that he made the connection himself in high school:

I think that was one of the things, last year, when we were working in calculus and then we started doing it in physics, I kind of put the two together. Um, cause we would talk about the average velocity and it — we didn’t get into the average velocity in, um, position and acceleration in math until the very end of the year. Well, not very end, but more towards the end. We learned most of this stuff in the beginning of physics. So I guess, kind of in my head, I put the two together. Change in speed type thing. I usually associate velocity with speed. And... just average rate of change. I mean, the speeds change — er, um velocity is changing so you can use that for anything that’s changing.

Paul also uses the physical context that problems were presented in to interpret his answers. I asked Paul to estimate the average rate of change between two points on a position versus time graph. Paul calculated his answer using the average rate of change formula and then stated: “So an average velocity of, um, -.6. So that kind of tells me that the object, over the interval, is moving, more in a backwards position from it’s starting point then it moves forward.”

Paul was able to compute average rates of change using a formula. He was able to apply a formula to find the average rate of change between data points listed in a table. Paul frequently wrote a general formula for average rate of change as $\frac{f(b) - f(a)}{b - a}$. In his work on the in-class calculus activities, Paul correctly computed average rates of change between $t$ and $t + h$ for various functions, including $f(t) = t^2$, $f(t) = t^3$ and $f(t) = t + r^2$. On the final examination, Paul correctly computed the average rate of change of $f(t)$ from $t = 1$ to $t = 3$ for the function $f(t) = 2t^2 - t$.

Finally, Paul worked competently with data presented in a tabular fashion. On the Average Rate of Change Pretest, Paul easily applied the average rate of change formula.
to compute the average velocities between different sets of times from a table showing
distance and time. The distance versus time table is shown below.

<table>
<thead>
<tr>
<th>t (sec.)</th>
<th>s (ft.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.0</td>
</tr>
<tr>
<td>0.2</td>
<td>0.5</td>
</tr>
<tr>
<td>0.4</td>
<td>1.8</td>
</tr>
<tr>
<td>0.6</td>
<td>3.8</td>
</tr>
<tr>
<td>0.8</td>
<td>6.5</td>
</tr>
<tr>
<td>1.0</td>
<td>9.6</td>
</tr>
</tbody>
</table>

The average velocity for the interval 0 is less than or equal to t which is
less than or equal to .2. Um, let’s see, for this one I….stuck with the
average rate of change trend. And…found the average between the two
points. I believe. So I took the change in, um, your, the change in feet
divided by the change in time, to give me 2.5…is the average velocity.
And the average velocity over the interval 0.4 to 0.8. Um, let’s see, for
this one, I used….um, I kind of disregarded the 6, the 0.6 and the 3.8 for
the coordinates for this one and I went straight from 6.5 to the 1.8 in the
interval given to us. And, over the given interval to give me [11.75].

Paul proficiently worked with average rate of change data presented in tables in many
other problems similar to the question presented above. Paul also used tables to organize
his own work on rate of change problems in the in-class calculus activities and on his
homework.

**Average Rate of Change Concept Image: Summary.** As Paul talked about
average rate of change during the interviews, he often incorporated both mathematics and
physics terminology into his discussions. Paul did not necessarily submerge the average
rate of change problems in a physics context, rather he used the physics language to help
him describe the meaning of average rate of change.

Paul has demonstrated his ability to work with average rate of change using a
variety of representations. Paul does not seem to exhibit a preference of working with
one representation over another for average rate of change type problems.
Derivative and Integral. As described previously, Paul did not rely on physics concepts to help him conceptualize the derivative or integral. Paul seemed to possess a balanced concept image of derivative and integral; that is, Paul solved competently solved problems using various representations of derivative and integral. Paul seemed to emphasize the graphic representation of derivative and integral in his interviews, homework assignments, and in-class activities. Consider Paul’s concept maps of Derivative and Integral (see Appendix E). Paul’s concept map of Derivative almost exclusively refers to graphical aspects of the derivative, such as the slope of the tangent line, maximums and minimums, and concavity.

Paul seemed to follow a pattern when solving problems where he was asked to sketch a graph of the derivative or an anti-derivative given the graph of a function. Paul’s process when sketching graphs of derivatives was first to identify the extrema of the function since the sign of the slope changes at those points. He then mapped the extrema points to the horizontal axis of the derivative graph and determined where the graph of the derivative was positive and negative based on the graph of the function.

Um, first I’m going to find...the maximums and minimums, so I can find...where the graph of the function is changing it’s slope so I know where the derivative is of it is – derivative is going to become positive or negative.

In another instance, Paul stated: “So I’d always go in and find, um…the...so here’s our original function. I’d find the minimum so I’d know where it [derivative] was going from positive to negative.”

Paul worked systematically when he produced an anti-derivative graph given a graph of a derivative function. As you will see from the following passage, Paul uses his
knowledge that the derivative is the slope of the function to proceed through Derivative Task 3, pictured in Figure 31.

Figure 31: Paul's Solution to Derivative Task 3
Paul was given the graph of $g'(x)$ and asked to sketch the graph of $g(x)$.

OK. I’m just going to go and find where the...slope is going to be positive and negative. Yeah, by identifying where the derivative is positive and when it’s negative- and it goes back to positive. At (0,0) it has...on the derivative, there’s a point of inflection...Now the concavity is going...uh, I suspect the concavity might change where the, where there’s a point of inflection but it’s not going to affect, ah, the, whether the slope is positive......the original function always has a maximum or minimum when this crosses the x-axis. Because... Because...this [the derivative] is the slope of the original function, so this has a negative slope and then it changes to a positive slope at this point [approximately 1.25]. That’s where the minimum should be. (pause)...(mumbling)...So does this function have a double root at, at that point?

Paul identified places where the derivative crossed the x-axis and marked them as extreme points on the anti-derivative graph. Paul made an interesting connection to the symbolic representation when he asked if the function had a double root at $x = 0$.

Paul’s work on Derivative tasks 5.1, 5.2, and 5.3 also showed his ease in working with the graphic representation of derivative. In Derivative Task 5.1, Paul was asked to approximate at which point $f(t)$ was increasing at a rate of 2.5 units per unit increase in $t$. Paul immediately interpreted the question as asking about slope: “OK, so I read that as asking when is it going to have, when is it going to rise 2 and one half units for every time it goes over, um, one unit t. So, so basically, when is it going to have a slope of, of two and a half.” It is evident from this passage that Paul was thinking about a function increasing in terms of slope. Furthermore, Paul continued to use the idea of rise over run to calculate slope.

Paul exhibited some misconceptions about motion that eclipsed his understanding of calculus. Consider the examination question and Paul’s answers pictured in Figure 32. For the velocity vs. time plot, Paul wrote, “At first the student began and had decreasing

A student walks beside a 2-meter measuring stick, beginning her walk at the origin. Then she moves with decreasing speed toward the 2-meter mark. After coming momentarily to rest near the 2-meter mark, the student immediately begins moving toward the 0-meter mark with increasing speed. For each of the plots below, sketch graphs of this motion and briefly explain why you drew the plots as you did.

**Brief Explanation:**

The student is approaching a distance of a decreasing speed, so the student begins to move with the rate it is approaching after coming momentarily to rest near the 2-meter mark. The maximum value on the graph represents where the student turns.

**Position vs. time**

At first, the student began with a decreasing velocity, which eventually reversed direction, increasing its velocity. So the line with decreasing slope shows that the decreasing velocity and the positive slope shows that the student has an increasing velocity. The very sharp turn represents the students turn.

**Velocity vs. time**

As the student approaches the 2-meter mark, it begins to move at a constant velocity, which is the average velocity between the two points. The graph splits at the point where the student obtains a positive acceleration.

**Acceleration vs. time**

Figure 32: Paul's Solution to a Kinematics Examination Problem

velocity and then stopped and reversed direction increasing its velocity. So the line with decreasing slope shows that the decreasing velocity and the positive slope shows that the student has an increasing velocity. The very sharp turn represents the students turn.”
Paul did not take into account the change in direction on his graph, even though he mentioned it in his explanation. It could be the case that Paul was viewing the graph that he drew as representing a change in direction: moving southeast to moving northwest. Paul did not account for the negative slope of his position versus time graph.

Paul's explanation for his acceleration versus time graph says, "As the student travels towards the 2m mark, it is decelerating so it has a negative acceleration, but the student turns around where the graph splits and it obtains a positive acceleration." Here Paul exhibited a classic motion mistake, confusing negative acceleration with slowing down (McDermott, van Zee, etc. 1987). Although Paul's acceleration vs. time graph was consistent with his velocity vs. time graph, his velocity vs. time graph did not match up with his position vs. time graph. It seemed that his physics misconceptions dominated his knowledge about derivatives. For on the same examination, Paul correctly sketched functions for acceleration and position given a velocity function. Talking about his solution to this graph problem, Paul stated:

First, so when I was given the velocity, on the test, I found where it crossed the x-axis and it's maximums and minimums. And then the next thing I did from there was to draw the graph of the acceleration or the derivative of velocity and where from positive to negative, this is where I had the derivative or the second derivative crossed the x-axis. Um, where I had a change in concavity in the velocity graph I had a minimum in my acceleration graph.

Notice that Paul spoke about features of the graph, such as maximums and minimums and concavity to describe his method of solution.

Paul demonstrated his ability to solve derivative and integral problems using the symbolic representation. For instance, Paul showed proficiency calculating derivative using the Power Rule, Product Rule, Quotient Rule, and Chain Rule on his homework.
assignments, in-class calculus activities, and examinations. The only area that Paul seemed to have difficulty taking derivatives involved exponential functions. Paul showed a pattern of difficulty applying the chain rule to exponential functions. For instance, on an examination, Paul differentiated $3^t$ as $6t$. Other than these minor mistakes, Paul comfortably calculates derivatives using the rules for differentiation.

Paul also competently solved integral problems using formulas. Paul computed the anti-derivative of polynomial functions, logarithmic and exponential functions, and trigonometric functions using substitution on examinations, homework assignments, and in-class activities. Paul easily computed $\int 3x^2 \, dx$ during the third interview. Paul described his method of solution as follows:

So the anti-derivative of $3x^2$ is $x^3$ plus C. Because, um, you want to add one...to the exponent because when you take the derivative of something you’re subtracting the power by 1 and multiplying by the power out front. In this case, um, when you add 1 to the exponent, it’s the same as the number out front so you don’t have to have any fraction or anything. ‘Cause when you differentiate x cubed you get 3 x squared.

Notice that Paul related the idea of anti-differentiation to differentiation. He called upon his knowledge that taking a derivative of a polynomial reduces the power of the polynomial by one to justify his solution to the anti-derivative problem.

On occasion, Paul would talk about graphical contexts in terms of algebraic formulas. For example, in the next passage, Paul was taking the derivative of a piece-wise defined function that is defined at $f(t) = 6.25$ for $A < t < B$ and $f(t) = 1.10t$ for $B < t < C$. Paul said:

So from A to B, um, we have a negative, about negative 6 and ¼ or something like that. And the derivative of a constant is zero. And it has no slope – or it has...zero slope. (pause – drawing). And from B to C, um it’s a linear function, so that would kind of be of the form $y = mx+b$. The
b term is going to drop out and the derivative is going to end up being, um, \( m \). Because we know by the...um...chain rule, or, I don’t remember the other one, by the chain rule, we’re going to drop \( x \), the power on the \( x \) by 1. So it’s going to be \( x \) to the zero, so that’s come to one and then you’re left with the constant, which is the slope. Which is one-ish.

Paul used his knowledge that the derivative of a constant is zero and the derivative of a linear function is a constant to help him solve this problem.

Paul correctly solved most derivative and integral numeric representation on homework assignments, examinations, and in-class activities. For example, Paul was able to calculate the derivative of a composition of functions from function data presented in a table and computed lower and upper estimates for the distance an object traveled from a velocity versus time table. Paul did not appear to convert tabular data into a graph or other representation. Rather, Paul appeared comfortable working with data directly from a table.

Finally, Paul exhibited an intuitive understanding of the Fundamental Theorem of Calculus. He often referred to taking derivatives as “working forwards” and taking antiderivatives as “working backwards”. Paul’s use of the ‘forward’ and ‘backward’ terminology to describe derivatives and integrals suggests that he conceptualized the two concepts as inverses of each other. Notice Paul’s use of the ‘forward’ and ‘backward’ vocabulary in his concept maps of derivative and integral (see Appendix E). On his concept map of Integral, Paul wrote, “Derivative of Integral of a function = the function”. This statement is indicative of Paul’s understanding of the Fundamental Theorem of Calculus.
**Derivative and Integral Concept Image: Summary.** Paul did not regularly use physics to help him solve or discuss derivative and integral problems. Paul proficiently worked with derivatives and integrals in a variety of representations. Paul appeared to possess a conception of derivative and integral that included graphic, symbolic, numeric and physical representations. Paul also demonstrated an intuitive understanding of the Fundamental Theorem of Calculus. Paul exhibited some physics misconceptions that interfered with his understanding of the derivative and integral concepts.

**Summary**

Paul was classified as a Language-Mixer in the Average Rate of Change category and a Non-User in the Derivative and Integral category. Paul blended physics and mathematics vocabulary as he talked through his solutions to average rate of change problems. Paul’s concept image of average rate of change was balanced, although Paul did not readily make connections between the average rate of change formula and the slope of the secant line early in the Fall 2000 semester.

Paul was classified as a Non-User in the Derivative and Integral category because he did not depend on physics concepts or examples to help him conceptualize derivative and integral problems. Paul proficiently solved derivative and integral problems in various representations. Paul’s concept maps along with his homework and in-class activities suggest that Paul preferred the graphical representation of derivative and integral.
Background

Jason is an Electrical Engineering major who has always been interested in computers and robotics. Jason reported that he decided to enroll in the Calculus/Physics course because he liked the idea of being able to combine concept and making explicit connections between the two subjects. He thought the integrated nature of the course would add an aspect to the class that he would not get if he enrolled in separate calculus and physics classes.

During the Fall semester, Jason enrolled in Environmental Ethics and Introduction to Electrical Engineering, as well as the Calculus/Physics course. Jason enrolled in Calculus/Physics, Introductory English, and Introduction to Scientific Programming in the Spring semester.

Jason indicated that although he was not involved in any clubs or organizations on campus, he enjoys music, playing guitar, and mountain biking. He hopes to spend a semester abroad in Germany, studying German engineering.

Jason enrolled in a year-long Advanced Placement calculus class during his senior year of high school. Jason reported that he studied limits, derivatives, antiderivatives, trigonometric functions, and logarithmic functions in his high school calculus class. Jason also enrolled in an honors physics class during his junior year of high school. He recalled discussing forces during his high school physics class.
Jason received a B in Calculus during the fall 2001 semester. Twenty five out of 48 students in the Calculus/Physics class received a grade in the range of B- to B+. Jason received a B- in the fall semester of Physics.

Physics Use Classification

Overview. Jason was classified as a Language-Mixer in the Average Rate of Change category and a Non-User in the Derivative and Integral category. Jason was classified as a Language-Mixer in the Average Rate of Change category because he blended mathematics and physics vocabulary as he talked though his solutions to average rate of change problems.

Jason did not appear to use physics in a concrete way to help him conceptualize derivative and integral problems. Thus, Jason was classified as a Non-User in the Derivative and Integral category. Jason’s methods of solution and language as he solved derivative and integral problems were strictly mathematical.

The next two sections present evidence for Jason’s classification as a Language-Mixer in the Average Rate of Change category and a Non-User in the Derivative and Integral category.

Average Rate of Change: Language-Mixer. On his Average Rate of Change Pretest, Jason indicated that when he heard the word ‘rate’, he pictured ‘one variable proportionally related to another’. During the first interview, Jason expanded on his written response:

Basically what I meant by one variable proportionally related to another was, ah, that you essentially have a number that represents something and it’s related to a number, another number, proportionally, depending on…ah, what’s involved. For example, speed, like miles per hour. Um…like if I was going 60 miles an hour, I’d be going 60 miles proportionally to that 1 hour. So every mile, every hour I’d be going 60
miles. And then if you increased it to 65 miles per hour, you’re going to increase the proportion from 65 to 1.

Notice that Jason used the example of miles per hour to explain what he meant by his answer to the question. Jason sometimes evoked physical examples and images when answering questions during the interviews, but he did not use the examples to help him solve average rate of change problems. Rather, he mentioned physics experiences or examples during his discussions of average rate of change. Jason often used a mixture of physics vocabulary and mathematical terminology as he answered questions and worked through problems. For example, when I asked Jason to explain what the expression \( f(x_1) - f(x_0) \) meant to him, he used a combination of mathematical and physics terminology to describe the expression. “This was just, this is just the…change in the distance on, on the vertical axes from one point to another. And it’s just the function of one point, \( x \) primary, minus, \( x \) initial, which is the, the first point. So you just find the change in distance, the change in the value from one point to another.” Notice that Jason used the words ‘change in distance’ and ‘change in value’ to describe the expression \( f(x_1) - f(x_0) \). The phrase ‘change in distance’ has a physical connotation, whereas the phrase ‘change in value’ has a mathematical undertone.

Jason talked about learning about average rate of change in his high school physics class. For example, I asked Jason how he knew that the slope of the secant line was the average rate of change and he answered, “Because…I was told so in physics class!” Jason never justified why it made sense to him that the slope of the secant line was the average rate of change; rather, Jason claimed that he was “going on faith”.

Finally, when I asked Jason to derive a general rule for finding the average rate of change between two points on a graph (Average Rate of Change Problem 8), he said,
“You just use — it would be like plugging it into the physics, physics formula. It would just be f(b) minus f(a) over b − a.” Even though the question was framed in mathematical terminology, Jason talked about the average rate of change formula as a ‘physics formula’.

**Derivative and Integral: Non-User.** Jason did not appear to rely on his knowledge of physics to help him solve derivative and integral problems. Jason tended to use strictly mathematical language as he solved both derivative and integral problems. During the second interview, I asked Jason if he ever thought about physics concepts as he worked through problems asking him to sketch derivative and antiderivatives. Jason replied, “I think it depends on what I’m doing. Like, when I just get graphs like this [Derivative Problems 1-4] that are just pictures of the graphs and I have to work one way or the other, I just usually try to go by the rules.” Jason seems to work through graphical derivative and integral problems using properties of graphs, physical derivative and integral problems using properties of physics, and symbolic derivative and integral problems using formulas. In a forthcoming section, I discuss Jason’s concept images of derivative and integral.

Finally, in his concept map of derivative and integral, Jason did not connect any physics concepts or terminology to the main concept of derivative and integral. A major portion of his concept map of derivative and integral is devoted to ‘the rules’ or methods of differentiation and integration. Jason’s concept map provides further evidence for his classification as a Non-User in the Derivative and Integral category.

**Summary.** Jason interspersed physics and mathematics vocabulary as he solved average rate of change problems. Jason’s past experiences with average rate of change,
particularly his experiences in his high school physics class, seemed to have an impact on
his conceptualization of average rate of change.

Jason did not use physics to help make sense of the derivative and integral
concepts. Rather, Jason paid particular attention to the representation that a problem was
presented in and used cues from that particular representation to solve the problem.

In the next section, I discuss my interpretation of Jason’s concept images of
average rate of change, derivative, and integral.

Concept Image

Overview. In this section, I will discuss Jason’s concept images of average rate of
change, derivative, and integral. I attempted to re-construct Jason’s concept images by
using his concept maps as well as his responses to interview tasks, homework
assignments, examinations, and in-class activities.

Average Rate of Change. Jason was very comfortable with the idea that the
average rate of change is the slope of the secant line. He primarily talked about the slope
of the secant line as he solved Average Rate of Change Problems 5 – 8. For example,
consider Jason’s explanation as he began to solve Average Rate of Change Problem 5:

Um…what is the average rate of change between…all right. So they want
to know the rate of change between x =1 and x = 2 on the graph. And, ah,
so what I’ll do…average rate of change, so I’ll draw a line between the
two, a secant line and then, ah, I just have to find the slope of that line to
find the average rate of change.

Jason’s first reaction was to draw the secant line connecting the points (1, 1) and (2, 4) on
his graph. Then Jason calculated the slope of the line connecting the points to figure out
the average rate of change. Jason determined the answer to be 3 and stated, “…and
there’s no units so it’s just going to be 3.” When I asked Jason why his answer of 3 made
sense to him, he replied, "It makes sense because, um, you draw this line, the slope of this line is 3 so any point along here – the slope is always going to be 3 so it's always changing by a factor of 3 or by a rate of 3."

Jason used the slope of the secant line to solve Average Rate of Change Problems 6, 7, and 8, as well. As he began to solve Average Rate of Change Problem 7, Jason said, "Negative 3 and 3. This is just a straight line so it's just going to be zero because the slope of that line is zero. I can do it out, but I'm lazy!" By 'doing it out', Jason was referring to plugging the ordered pairs (-3, 9) and (3, 9) into the average rate of change formula and calculating the answer to be zero.

Jason sometimes evoked graphical images when talking about average velocities. For example, when I asked Jason how he knew he could use the average rate of change formula to calculate average velocity, he claimed:

Because average velocity is going to be a rate, it's going to be in units of measure per unit of time. So, um, this essentially would correspond to a graph. It would be like drawing the secant line between these two sets of points and then using the formula from before [Average Rate of Change Problems 5 – 8] and drawing a line and finding the slope of the line.

Notice that Jason justified his use of the average rate of change formula by equating his method of solution to constructing the secant line and calculating the slope of the line. The preceding passage presents evidence of Jason's well connected notions of average rate of change and slope of the secant line.

Jason seemed very comfortable with the notion that average velocity was an average rate of change. When I asked Jason during the first interview how he knew he could use the average rate of change formula to find average velocity in Average Rate of Change Problem 9.1, he said, "Um, because average velocity is going to be a rate, it's
going to be in units of measure per unit of time.” Jason then went on to connect the average rate of change formula to the slope of the secant line by saying, “This would essentially correspond to a graph. It would be like drawing the secant line between these two sets of points and then using the formula form before and drawing a line and finding the slope of the line.”

Jason also competently solved average rate of change problems using the symbolic representation. On the in-class calculus activities, Jason correctly computed average rates of change between $t$ and $t + h$ for various functions, including $f(t) = t^2$, $f(t) = r^3$, $f(t) = t + r^2$, and $f(t) = 2^t$. On the final examination, Jason correctly computed the average rate of change of $f(t)$ from $t = 1$ to $t = 3$ for the function $f(t) = 2r^2 - t$.

Jason felt comfortable using the average rate of change formula and recognized that the average rate of change formula was the same as the formula for calculating the slope of the secant line. For example, as Jason solved Average Rate of Change Problem 10, he said, “And I’m just going to be finding the slope of that line [between (1, 0.7) and (3, -0.5)], so I use the same formula, which is, uh, the function of $x_1$ minus the function of $x_0$ over $x_1$ minus $x_0$.”

Jason worked well with most average rate of change problems in a numeric representation. He easily applied the average rate of change formula to find the average velocity of an object from a position versus time table on homework assignments, in-class activities, and interview tasks.

**Average Rate of Change: Summary.** Jason tended to use both physics terminology and mathematical vocabulary as he talked about his solutions to the Average Rate of Change problems. Jason comfortably shifted between images of rate as slope of
the secant line, as a unit of measure per unit time, and as the formula \( \frac{f(b) - f(a)}{b - a} \) as he solved average rate of change problems. Jason's shifts between different images were manifested in his mixing of the physics and calculus vocabularies.

While Jason demonstrated that he held a balanced concept image of average rate of change through his ability to work with average rate of change in multiple contexts, Jason's image of average rate of change as slope of the secant line seemed to dominate other representations. Many of his responses to interview questions, as well as his justifications of his work on examinations and homework assignments included a description of the average rate of change as the slope of the secant line.

**Derivative and Integral.** As previously discussed, Jason did not appear to rely on physics concepts to enhance his understanding of the derivative and integral. Jason comfortably used multiple representations when solving derivative and integral problems.

Jason showed an understanding of the graphical relationship between a function and its derivative. During his second interview, Jason commented, "The derivative is the slope of the graph [of a function] at all its points." However Jason indicated that he thought about the relationship between a function and its derivative in a formulaic manner. Jason commented, "Cause when you take the derivative, it's always one degree less." I believe that Jason's comment referred to his experiences with taking derivatives of polynomial functions. At the time of Jason's second interview, the Calculus/Physics class had focused primarily on polynomial functions.

As he solved derivative tasks during his interviews, Jason described his process of approaching graphical derivative problems. For instance, consider Jason's answer to Derivative Task 1:
All right. Well, I guess first I’d label the maximums and minimums [of the function, \( f(t) \)]. Cause that would be those points where the derivative is equal to zero. Cause the slope of the graph at those points is equal to zero. And then, ah, I’d look at the approximate slope between different points.

Jason’s process for solving derivative and anti-derivative graph problems began with his identification of the maximums and minimums of the function. While working on the same problem, Jason discussed the shape of the graph and how he used the shape of the function graph to construct the graph of the derivative. Jason’s work is pictured in Figure 33.

I’ll start with the left endpoint to A. The slope [of the tangent lines] is pretty much – it starts out incredibly positive because the graph is very steep there and then it kind of levels off, it gets less positive. So it starts out…positive, extremely positive at first and then it hits A, where it becomes zero. And after A it becomes negative. It starts kind of negative at first and then it gets more and more negative and then goes back the other way, towards B, where it becomes zero.

Notice Jason’s descriptions of different degrees of positive and negative slopes. Jason described the slope of the graph near the y-axis as ‘extremely positive’. Likewise, Jason says the graph is ‘kind-of negative’ after the point A.

Jason demonstrated his ability to solve derivative problems presented to him in graphical contexts in his homework assignments, in-class activities, and examinations. Furthermore, in his concept map of derivative, Jason included the notions of slope, concavity, and ‘increase/decrease’ as concepts indirectly connected to the central concept of derivative.

Jason also comfortably sketched graphs of anti-derivatives given the graph of a function. In fact, Jason exhibited a tendency to sketch the graph of an antiderivative to help him solve certain problems. For instance, consider Jason’s solution to the in-class
activity pictured in Figure 34. Notice that Jason sketched the graph of the antiderivative to answer the questions, rather than estimating area under the curve. This example
1. The graph of \( f'(t) \) is given below. Determine which of the points \((f_0, f_1, f_2, f_3, f_4, f_5, f_6)\) satisfy the following properties:

(a) \( f(t) \) is greatest

(b) \( f(t) \) is least

(c) \( f'(t) \) is greatest

(d) \( f'(t) \) is least

![Graph of \( f'(t) \).](image)

**Figure 34: Jason's Solution to an In-Class Integral Activity**

should not lead one to believe that Jason did not understand the significance of area under the curve. For example, Figure 35 shows Jason's work on an examination problem that asked him to find the change in momentum from a force versus time graph. Notice that Jason computed the area for the first two graphs by counting and adding unit blocks under the curves.
7. (9 both pts) Find the change in momentum from \( t = 0 \) to \( t = 4 \) for each of the following graphs.

\[
\begin{align*}
2 - q &= -2 \, \text{Ns} \\
3 - q &= -1 \, \text{Ns} \\
\int_0^4 -0.5t^2 + q = -5 \, \text{Ns} \\
\int_0^4 -0.5t^2 + q = \frac{5}{3} (q(4)^3 - q(q)) \\
\frac{26}{3} \, \text{Ns}
\end{align*}
\]

Figure 35: Jason’s Solution to a Change in Momentum Examination Problem

Jason often used the physical parameters and constraints given in a problem to help him solve the problem. For example, consider Jason’s answer to the Calculus/Physics examination problem is pictured in Figure 36. Notice that Jason’s graphs both model the motion described in the problem and are consistent with one another (position with velocity and velocity with acceleration). Furthermore, Jason’s explanations accompanying his graphs clearly describe the motion of the student in the problem and justify why he drew his graphs as he did.

Jason’s work on the in-class activity pictured in Figure 37 also demonstrates his use of physical constraints in derivative and integral problems. Notice that Jason justified
A student walks beside a 2-meter measuring stick, beginning her walk at the origin. Then she moves with decreasing speed toward the 2 meter mark. After coming momentarily to rest near the 2 meter mark, the student immediately begins moving toward the 0 meter mark with increasing speed. For each of the plots below, sketch graphs of this motion and briefly explain why you drew the plots as you did.

**Brief Explanation:**

- **Position vs. time:**
  - She starts walking away quickly while slowing down, then stops and heads the other direction while increasing speed back toward the origin.

- **Velocity vs. time:**
  - The velocity is steadily decreasing till she stops (t), then becomes negative as she heads back toward the origin.

- **Acceleration vs. time:**
  - She has a constant negative acceleration as she is continually slowing down till (t), then speeding up in a negative direction after (t).

Figure 36: Jason's Solution to a Kinematics Examination Problem

his answer not by comparing areas under the two curves, but by invoking arguments based on velocity. Jason demonstrated his ability to use properties of physical systems to...
3. Two cars start from rest at a traffic light and accelerate for several seconds. The following graph shows their velocities vs. time. [3, p. 308]

![Graph of car velocities vs. time]

(a) Which car is ahead after one second? How do you know? 
Car 2, its velocity has been greater so it has gone further.

(b) Which car is ahead after two seconds? How do you know? 
Car 1 because it increased its velocity more than Car 2 decreased. Therefore it has pulled ahead.

Figure 37: Jason’s Solution to an In-Class Calculus Activity

justify his work on other classroom activities as well as homework assignments and examinations.

Typically, Jason competently solved most derivative and integral problems in a symbolic representation. Jason computed anti-derivatives of polynomial functions, logarithmic and exponential functions, and trigonometric functions using substitution on
homework assignments, examinations, and in-class activities. However, Jason did not seem comfortable computing integrals using the inverse trigonometric functions. Additionally, Jason sometimes confused the operations of differentiation and anti-differentiation on homework assignments and examinations. Additionally, Jason did not seem confident solving implicit differentiation problems. Jason frequently avoided solving implicit differentiation problems or attempted to isolate one variable of the equation on his homework assignments, in-class activities, and examinations.

Jason solved most derivative and anti-derivative problems in a numeric representation with ease. For example, Jason correctly computed the derivative of a composition of functions from a table of various function values on homework assignments and in-class activities. Jason also computed lower and upper estimates for the distance an object traveled from a velocity and time table. Jason rarely sketched out graphs to help him solve numeric integral problems. Rather, Jason worked well with data presented to him in a tabular fashion. Consider Jason’s work on the classroom activity pictured in Figure 38. Notice that Jason computed upper and lower estimates of the velocity at t = 5 using only the data in its numeric format.

Finally, Jason rarely mentioned the Fundamental Theorem of Calculus during any of the interviews or in his responses to homework problems, in-class activities, or examination problems. Although Jason drew one concept map for derivative and integral he did not mention the Fundamental Theorem of Calculus on his concept map. Jason connected the two central concepts of derivative and integral with ‘functions’ and ‘variables’. Jason also did not talk about the integral or antiderivative as an inverse derivative operation as some other students in the present study talked about the integral.
2. The following table gives the acceleration, \( a \), in m/sec\(^2\), after \( t \) seconds of jumping out of an airplane. [3, p. 157]

<table>
<thead>
<tr>
<th>( t ) (sec)</th>
<th>( a ) (m/sec(^2))</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>9.81</td>
</tr>
<tr>
<td>1</td>
<td>8.03</td>
</tr>
<tr>
<td>2</td>
<td>6.53</td>
</tr>
<tr>
<td>3</td>
<td>5.38</td>
</tr>
<tr>
<td>4</td>
<td>4.41</td>
</tr>
<tr>
<td>5</td>
<td>3.61</td>
</tr>
</tbody>
</table>

(a) Give upper and lower estimates of your velocity at \( t = 5 \).

\[
\begin{align*}
\text{Upper} &= 9.81 \times 1 + 4.41 \times 1 + 5.38 \times 1 + 6.53 \times 1 + 8.03 \times 1 = 37.77 \\
\text{Lower} &= 9.81 \times 1 + 4.41 \times 1 + 5.38 \times 1 + 6.53 \times 1 + 8.03 \times 1 = 27.16
\end{align*}
\]

(b) Get a new estimate of your velocity at \( t = 5 \) by taking the average of your upper and lower estimates.

\[
\frac{37.77 + 27.16}{2} = 32.865 \text{ m/sec}
\]

Figure 38: Jason’s Solution to a Numeric Integration Problem

**Derivative and Integral Concept Image: Summary.** Jason did not regularly use physics to help him solve or discuss derivative and integral problems. Jason proficiently worked with derivatives and integrals in a variety of contexts. Jason appeared to possess a conception of derivative and integral that included graphic, symbolic, numeric, and physical representations.

**Summary**

Jason was classified as a Language-Mixer in the Average Rate of Change category and a Non-User in the Derivative and Integral category. Jason blended physics and mathematics vocabulary as he talked through his solutions to average rate of change...
problems. Jason’s concept image of average rate of change was balanced, although Jason demonstrated a preference for using the graphic representation of average rate of change.

Jason was classified as a Non-User in the Derivative and Integral category because he did not depend on physics concepts or examples to help him conceptualize derivative and integral problems. Jason proficiently solved derivative and integral problems in various representations. Jason often used the physical parameters and constraints of derivative and integral problems to help him solve the problems.

Summary

This chapter presented the results of applying the Physics Use Classification Scheme to the data collected for each of the eight students. Additionally, my interpretation of each student’s concept image of average rate of change, derivative, and integral were presented and discussed. In the next chapter, I will summarize the major results of the present research study as well as discuss directions for future research.
CHAPTER VI

DISCUSSION AND CONCLUSIONS

Overview

The preceding chapters have presented the theoretical framework, research methodology, and data on the students' use of physics to inform their conceptualizations of average rate of change, derivative, and integral. This chapter will tie these components together by presenting a summary and discussion of the major results of the data analysis.

The data analysis was conducted on two levels: a microanalysis and cross-case analysis. The microanalysis of the data yielded a classification scheme that describes how the students use physics to inform their conceptualizations of average rate of change, derivative, and integral. The Physics Use Classification Scheme was tested and refined during a Within-Case analysis. Recall that the four categories in the Physics Use Classification Scheme are Contextualizers, Example-Users, Language-Mixers, and Non-Users. The Physics Use Classification Scheme will be discussed in more detail in this chapter in order to highlight and summarize the distinctions between the categories. In discussions with other researchers about the Physics Use Classification Scheme, a need for a way to rank the categories arose. In order to address the need to order or rank the Physics Use Categories, a continuum was developed that ranked the categories according to the level of use of physics exhibited by students in each category. The
continuum used to rank the categories is called the Abstract/Concrete Continuum and will be discussed in more detail later in this chapter.

The two major themes that I uncovered during the Cross-Case analysis were: (1) Students participating in the present study have an understanding of calculus concepts deeper than what has been previously reported in the literature. (2) The students tended to use physics concepts less often to help them solve derivative and integral problems than average rate of change problems. Furthermore, the cross-case analysis led to the development of a hypothesis regarding the difference in physics uses between average rate of change and derivative and integral. The results of the Cross-Case analysis will be discussed in more detail in a forthcoming section.

Physics Use Classification Scheme

In Chapter V, I discussed the development of the Physics Use Classification Scheme from the microanalysis. As a result of the microanalysis, four categories emerged and were refined during the other stages of analysis. The emergent classification categories are Contextualizers, Example-Users, Language-Mixers, and Non-Users. These classifications refer to the manner in which the students use physics concepts to aid in their conceptualization of calculus concepts. I will discuss the Physics Use Categories in order to summarize, compare, and contrast the categories as well as to remind the reader of the categories in light of a discussion about the organizing Abstract/Concrete Continuum.

The four Physics Use Categories emerged because I observed marked differences in the way certain students were using physics to help them solve calculus problems. For
example, although Rob and Terry both use physics as they solve derivative and integral problems, their use of physics is considerably different. When Rob solved Derivative Task 3, he stated:

Um, if there was a velocity, it would be slowing down, so it’s decreasing. I can reflect that by showing...this starting somewhere...(pause)...It would, position would still be increasing and then it goes negative once it gets here because, it’s a negative velocity for that time. So you’d probably want to go down and then go up [on the graph of the antiderivative].

Terry, on the other hand, approached the problem very differently:

This is increasing--for the first interval, it's got to be increasing, because the G prime is positive, so the slope of the tangent on this function is going to be positive. And it looks like it's getting--the slope's getting smaller, so it's probably like this. It crosses--it goes through zero, zero, so the slope, it's got to be tangent at the y axis, and it looks like it goes negative for a bit, something like this, and then lots and lots of positive, very quickly. Something like that, I think.

Rob and Terry both use physics to solve Derivative Task 3, but the manner in which they use physics is extremely different. Recall that Rob was classified as a Contextualizer. Rob talked about the problem in terms of velocity and position — a completely physical representation. Rob solved the problem using the fact that if the object is slowing down then the graph of position is decreasing. Rob equates slowing down with negative velocity and negative velocity with decreasing position.

On the other hand, Terry’s use of physics terminology is very limited in the above passage. Recall that Terry was classified as a Language-Mixer. His only mention of physics in this passage occurs when he talks about the graph moving ‘very quickly’ toward the end of his discussion of the problem. Terry did not depend on his knowledge of physics concepts to help him solve this problem. Rather, Terry’s use of physics was limited to the use of some physics terminology to help him describe the graph. Terry
appeared to use mathematical properties of the graph to help him solve the problem. Notice Terry's use of the mathematical terminology 'slope' and 'tangent' as he talked through his solution process. This mathematical vocabulary is absent from Rob's discussion of the problem, indicating that Rob did not depend on the mathematical properties of the graph, but rather the physical properties of the graph, to help him solve the problem. Because of differences in the way the students used physics to solve the problems, I found it necessary to distinguish the differences in physics use. Below I present a description of the four Physics Use Classifications.

**Contextualizers:** Contextualizers not only discuss calculus problems in terms of physics, but show evidence of immersing problems in physical contexts in order to solve them. Contextualizers use physical representations to solve many calculus problems. For example, Rob, who was classified as a Contextualizer, used a physical representation to solve Derivative Task 1, which was not explicitly stated in terms of physics. Rob solved this problem by referring to a specific physical situation.

But to me it seems that...if it was a ball that you pushed across a table and it was, a distinct v...I'll label these t and these v...distance...so times time, distance is decreasing and then it stops, turns around, goes backwards, time is still going...so in terms of velocity, that could be positive.

Rob labeled the axes of the graph as distance (y-axis) and time (x-axis) and he also labeled the axes of his solution graph as velocity (y-axis) and time (x-axis). Rob used the axes labels to help him make sense of the physical situation. Notice he says, "...time is still going forward," referring to the fact that the ball rolls forward and backward, but time is steadily passing.
**Example-users:** Example-Users are those students who refer to examples from physics to help them make sense of calculus concepts and problems. They do not contextualize the problem, that is, they will talk about physics in a way that is disconnected from the problem at hand. They also tend to use physical phenomena to make sense of an answer to a calculus problem. For example, an Example-User will invoke the relationship between position and velocity to justify an answer to a derivative problem.

**Language-mixers:** Language-Mixers are those students who tend to use language from physics in their discussion of calculus concepts. They use a concrete, physical language to discuss problems without contextualizing the problem or referring to an example. For example, a Language-Mixer will use the physics term ‘average velocity’ and the mathematical ‘average rate of change’ to describe his/her work on an average rate of change problem.

**Non-users:** Non-Users are those students who simply do not use physics, in any sense, to help them conceptualize calculus concepts. These students’ discussions of calculus problems involve non-physical vocabulary. Non-Users rely on their conceptualizations of calculus concepts in order to solve calculus problems.

The categories in the Physics Use Classification Scheme were developed as a means to describe the students’ use of physics. It appears that the categories can be linearly arranged in a continuum that describes the level of concrete physics use representative of each of the Physics Use categories. Figure 38 is a picture of the Abstract/Concrete Continuum.
The headings on the top of the continuum (Less Concrete and More Concrete) represent a continuum of how students use physics in solving calculus problems. A student who is less concrete is comfortable working with abstract symbols and mathematical formulizations to solve calculus problems. A student who is less concrete is less committed to using physics to help solve calculus problems. That is, a 'less concrete' student does not have the need to use physics to help him/her conceptualize calculus problems. On the other hand, a student who is more concrete prefers to think about calculus problems in terms of physical situations in order to solve them. Thus, a student who is more concrete is more committed to using physics to help solve calculus problems. That is, a 'more concrete' student uses physics to help him/her make sense of calculus problems.

The Abstract/Concrete Continuum was developed to organize and rank the Physics Use Categories. As mentioned previously, a need for a way to organize the Physics Use Categories arose in discussions about the categories. In particular, the Abstract/Concrete Continuum will help serve as a guide for future research. For instance, a natural question arises as to the benefits of being labeled in one category versus another. Future research can begin to address questions of this sort by examining and
comparing the achievement and qualities of students who lie at different points on the Abstract/Concrete Continuum.

The Physics Use categories were placed on the Abstract/Concrete Continuum by evaluating the level of commitment to using physics to help conceptualize calculus concepts exhibited by the students in each Physics Use category. The terminology 'more concrete' and 'less concrete' along with the Abstract/Concrete Continuum were developed with the help of Dawn Meredith, Associate Professor of Physics at the University of New Hampshire. Her expertise in physics and physics education was used to help place the Physics Use categories on the continuum. Contextualizers, who use physics to help make sense of calculus problems by submerging calculus problems in a physical context, exhibited the most commitment to using physics to conceptualize calculus concepts. Thus, Contextualizers were placed closest to the 'More Concrete' endpoint of the continuum.

The Example-Users were placed closer to the 'More Concrete' endpoint than the Language-Mixers since the Example-Users tend to talk more about physical situations and examples as they solve calculus problems. The Example-Users use of physics, although not as prominent as the Contextualizers, still indicates a substantial level of commitment to physics. The Example-Users use of physics indicates that they are using physics to help them make a calculus problem more meaningful. On the other hand, the Language-Mixers use of physics vocabulary does not indicate that they are using physics to help them make a calculus problem more meaningful. The Language-Mixers use of physics tends to be for descriptive purposes only. That is, the Language-Mixers do not rely on their understanding of physics to help them solve calculus problems. Finally, the
Non-Users are not committed to physics at all. Thus, the Non-Users were placed closest to the ‘Less Concrete’ endpoint on the continuum.

The Abstract/Concrete Continuum was developed as a way to organize the Physics Use Classifications. The Abstract/Concrete Continuum does not place a value on the Physics Use Classifications; rather it organizes the classifications around the level of commitment to physics. I attempted to find evidence that students labeled in one category outperform students in other categories on certain calculus tasks. I noticed that the stronger students, namely Terry, Michelle, and Paul, who received a grade of A in Calculus during the Fall semester, were all classified at the ‘less concrete’ end of the continuum for the categories of Average Rate of Change and Derivative and Integral. However, Jason, who received a B in Calculus during the Fall semester, also was classified at the ‘less concrete’ end of the continuum for the categories of Average Rate of Change and Derivative and Integral. Rob, who was the weakest student in the group, received a B- in the Fall semester of Calculus. Recall that Rob was classified as a Contextualizer in the categories of Average Rate of Change and Derivative and Integral. However, Rob had taken only one semester of Calculus during high school whereas the other seven students all took yearlong Advanced Placement Calculus courses in high school. Since Rob had the least amount of formal (classroom) experience with calculus among the eight students, he may have been more inclined to rely on physics concepts and experiences to help him conceptualize the calculus concepts and problems. Future research could address the motivation for students to use physics in concrete ways in order to solve calculus problems and conceptualize calculus concepts.
Future research also could investigate if students labeled in one category perform better on certain calculus tasks or develop richer understandings of average rate of change, derivative, and integral. This research could look for any correlation between Physics Use Classification and overall performance in calculus, accounting for differences in students’ high school calculus and physics backgrounds, SAT scores, and any other mathematics pretest scores. Additionally, future research could investigate differences between the concept images of students labeled in different Physics Use categories.

Finally, although the Physics Use Classification Scheme was used to classify the eight students in the present study, I intend to further modify the classification scheme and further clarify the descriptions of each category in future research. Future research could also investigate if all students classified in one category solve specific calculus problems in a certain way. For instance, future research could examine if all Contextualizers solve graphical derivative problems by immersing them in a position-velocity context.

Cross-Case Analysis Results

The purpose of the cross-case analysis was to uncover themes common to all cases. Specifically, I looked at comparing students’ performances on interview tasks and selected homework assignments, examinations, and in-class activity problems. My goal in performing the cross-case analysis was twofold: (1) To identify characteristics of the eight students’ understanding of calculus concepts and compare these characteristics of their understandings to descriptions of students’ understandings of calculus concepts.
previously discussed in the literature. (2) To generate hypotheses about the eight students’ use of physics. The Cross-Case analysis yielded three major findings that I will list and discuss in detail.

Comparison to Previously Reported Results

Students participating in the present study have an understanding of calculus concepts deeper than what has been previously reported in the literature. The eight students in the present study competently worked with the concepts of average rate of change, derivative, and integral in a variety of contexts. I will discuss the results of the present study in light of the existing literature in three parts: Average Rate of Change, Derivative, and Integral.

Average Rate of Change. Bezuidenhout (1999) reported that many students she investigated confused the notions of average rate of change and arithmetic mean. That is, the students in Bezuidenhout’s study allowed their conception of arithmetic mean to dominate their understanding of average rate of change. In the present study, Rob initially exhibited the same behavior that Bezuidenhout (1999) reported. Early in the semester, Rob attempted to use the arithmetic mean to compute the average rate of change. However, by the end of the semester, Rob’s notion of arithmetic mean no longer dominated his understanding of average rate of change.

During the first interview with each of the students, I administered a task adapted from Bezuidenhout’s 1999 study (see Appendix A – Average Rate of Change Problems 5.1, 5.2, and 5.3). When Bezuidenhout administered the task to 100 South African first year calculus students, 54% answered Task 5.1 correctly, 30% answered Task 5.2 correctly, and 26% answered Task 5.3 correctly. In the present study, 5 out of 8 (62.5%)
students answered Task 5.1 correctly, although the three students who did not answer Task 5.1 correctly all mentioned slope in their discussions of the problem. Five out of the eight students in the present study also answered Tasks 5.2 and 5.3 correctly. None of the students attempted to simplify the expressions in Tasks 5.2 and 5.3 algebraically. Bezuidenhout found that some students in her 1999 study attempted to solve Tasks 5.2 and 5.3 algebraically.

Orton (1984) stated that many students in his study confused the notions of average and instantaneous rates of change. Students in the present study did not seem to exhibit a confusion between average and instantaneous rates of change. In fact, six out of the eight students in the present study distinguished between average and instantaneous rates of change on their Rate of Change concept maps.

**Derivative.** Previous research concluded that students are adept at carrying out computational differentiation tasks, but have difficulty working with derivatives in other contexts (Orton, 1983; Ferrini-Mundy & Graham, 1994; Aspinwall, Shaw, & Presmeg, 1997). In particular, research on students’ understanding of derivative has shown that students could not explain the relationship between a function and its derivative or how the tangent line relates to the derivative. Students in the present study tended to define the derivative as the slope of the tangent lines at each point of the function. For example, Paul said, “Always I look at the derivative as the slope of the original function.” Terry said, “In terms of how the tangent is, that’s all that goes through my head, is where the tangent would lie on the function, whether it’s positive or not; that puts the f prime graph, then, either above or below the x axis, and that’s pretty much how I think about it, in terms of graphing it.”
Students in the present study used the tangent line or other properties of graphs to describe the relationship between a function and its derivative. For example, Michelle described the relationship between a function and its derivative as, “Above and below [the horizontal axis] of the derivative means increasing and decreasing of the function.” Rob used physics examples to help him describe the derivative: “Well a function, I just see it as, um, the physics part, I guess. Um, the position is when you take the derivative of it you get the velocity, when you take the derivative of that you get the acceleration.”

Students in the present study not only successfully solved computational derivative problems, but they also appropriately solved derivative problems in graphical, physical, and numeric contexts. Many students defined the derivative in terms of the slope of the tangent line. Five out of eight students connected the slope of the tangent line to the derivative concept in their concept maps of Derivative.

Integral. Previous research concluded that students have difficulty understanding integration as the limit of an infinite sum (Orton, 1983) and some students interpret the integral as an indication to perform a task (Ferrini-Mundy & Graham, 1994). Many students in the current study recognized the connection between Riemann sums and the integral. Rob remarked, “All the integral is...from is, um, doing a Riemann sums. Which is where you just draw little rectangles. If you had a curve you draw rectangles and as the number of rectangles approached infinity that where it turns into...the....integral.” Travis indicated that he believed the integral gave a ‘more accurate’ answer than the Riemann sum.

Students in the present study identified the integral not merely as an indication of ‘something to do’, but as the area under the curve. Five out of the eight students
indicated that the integral was related to either area under the curve or Riemann sums on their concept maps of Integral.

Summary. The students in the present study exhibited a richer understanding of average rate of change, derivative, and integral than what has been previously reported in the literature. In particular, students in the present study did not confuse the notions of average and instantaneous rates of change. Many students in the present study also conceptualize the derivative as the slope of the tangent line of its function at every point. Most students define the integral as the area under a curve.

I believe the students in the present study exhibited richer understandings of average rate of change, derivative, and integral for a number of reasons. High school teachers and curriculum developers might be integrating results of past research. The exposure to an integrated, conceptually-focused calculus and physics curriculum directly influenced students' conceptualizations of calculus concepts. Future research should examine these hypotheses.

Abstract/Concrete Use of Physics

Recall that the students were classified using the Physics Use Classification Scheme in two categories: Average Rate of Change and Derivative and Integral. Each of the eight students received a separate Physics Use Classification in the areas of Average Rate of Change and Derivative and Integral because I noticed marked differences in the way that the students were using physics to solve Average Rate of Change problems as compared to their use of physics to solve Derivative and Integral problems. An interesting pattern emerged as I considered the students' classifications as presented in Table 7. Recall that Contextualizers use physics in the most concrete way to help
conceptualize calculus concepts and Non-Users use physics in the most abstract way to help conceptualize calculus concepts. (Refer to Figure 39 for a picture of the Abstract/Concrete Continuum.) As one reads Table 7 from left to right for each student, except for Travis, the level of concreteness decreases or remains the same as one moves from the Average Rate of Change Classification column to the Derivative and Integral Classification column.

<table>
<thead>
<tr>
<th>STUDENT</th>
<th>AVERAGE RATE OF CHANGE CLASSIFICATION</th>
<th>DERIVATIVE &amp; INTEGRAL CLASSIFICATION</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rob</td>
<td>Contextualizer</td>
<td>Contextualizer</td>
</tr>
<tr>
<td>Scott</td>
<td>Contextualizer</td>
<td>Language-Mixer</td>
</tr>
<tr>
<td>Terry</td>
<td>Language-Mixer</td>
<td>Language-Mixer</td>
</tr>
<tr>
<td>Todd</td>
<td>Example-User</td>
<td>Non-User</td>
</tr>
<tr>
<td>Michelle</td>
<td>Example-User</td>
<td>Non-User</td>
</tr>
<tr>
<td>Paul</td>
<td>Language-Mixer</td>
<td>Non-User</td>
</tr>
<tr>
<td>Jason</td>
<td>Language-Mixer</td>
<td>Non-User</td>
</tr>
</tbody>
</table>

Table 7: Physics Use Classifications

For example, Paul was classified as a Language-Mixer in the Average Rate of Change column and a Non-User in the Derivative and Integral column. Language-Mixers exhibit a more concrete use of physics than Non-Users. Thus Paul’s moved from using physics in a more concrete way to conceptualize derivative and integral problems to using physics in a less concrete way to conceptualize average rate of change problems. Notice that two students, Rob and Terry, had the same Physics Use classification for their work with Average Rate of Change and Derivative and Integral. A hypothesis was developed to explain the apparent shift in the students’ concrete use of physics. Students are more apt to conceptualize average rate of change problems more concretely with respect to physics because they have experienced average rate of change phenomena in their everyday lives. For example, most students have experienced the concept of average
velocity of a car during a trip or the average velocity of themselves as they run a race. Because the notion of average rate of change is more likely to be grounded in students’ everyday experiences, they will be more likely to commit to physics to help them solve average rate of change problems.

On the other hand, many students are not comfortable conceptualizing or talking about rate “at an instant”. Students do not typically consciously experience instantaneous velocity or instantaneous rates of change in their everyday lives. Since students experience instantaneous rates of change less frequently than they experience average rates of change, students do not have the same physical experiences to draw upon and thus are less committed to physics to help them solve derivative and integral problems. I intend to further test and refine the hypothesis in future research.

**Summary of Results**

The major question investigated in the present research study is: How do students draw upon physics concepts to inform their understanding of rate of change, derivative, and integral? I found that students use physics concepts in four different ways to inform their understanding of rate of change, derivative, and integral. Students were classified as Contextualizers, Example-Users, Language-Mixers, or Non-Users in the categories of Average Rate of Change and Derivative and Integral. In the category of Average Rate of Change, two students were classified as Contextualizers, two students were classified as Example-Users, and four students were classified as Language-Mixers. In the category of Derivative and Integral, one student was classified as a Contextualizer, one student was classified as an Example-User, two students were classified as Language-Mixers,
and four students were classified as Non-Users. An hypothesis about how everyday experiences and familiarity might account for students’ concrete use of physics to solve average rate of change problems was generated. This hypothesis should be investigated in future research.

The Physics Use Categories were organized along an Abstract/Concrete Continuum. The Physics Use categories were placed on the Abstract/Concrete Continuum by evaluating the level of commitment to using physics to help conceptualize calculus concepts students in each Physics Use category exhibited. Although there was not enough evidence in the present study to formulate an hypothesis about whether students labeled in one category perform better on calculus tasks or develop a richer understanding of calculus concepts, future research could address this issue.

The present research study also examined a series of sub-questions. Each question will be discussed below.

1. Do students’ misunderstandings of fundamental physics concepts misinform their understanding of calculus concepts? Results suggest that students’ misunderstandings of fundamental physics concepts sometimes interfere with their understanding of calculus concepts. Specifically, I found that students who exhibited a position-velocity, velocity-speed, or velocity-acceleration confusion tended to allow their confusion to dominate their work on derivative and integral problems presented in a physical context. For example, Rob held the misconception that if an object is speeding up then the object’s acceleration is greater than zero. Rob’s misconception caused him to incorrectly sketch an acceleration versus time graph from a velocity versus time graph. Other students in the study also exhibited similar misconceptions.
2. Do students consistently use physics in a certain way to help them understand calculus concepts? The eight students in the present study tended to use physics differently when solving average rate of change problems and derivative and integral problems. Typically, students used physics in a more concrete manner to solve average rate of change problems than they did to solve derivative and integral problems. A hypothesis was developed to account for the difference in the students’ use of physics when solving average rate of change and derivative and integral problems. The hypothesis states that the notion of average rate of change is more likely to be grounded in students’ everyday experiences, thus they will be more likely to commit to physics to help them solve average rate of change problems.

3. Do students in the present study possess conceptualizations of calculus concepts similar to those of students previously documented in the literature? The students in the present study exhibited a richer understanding of average rate of change, derivative, and integral than what has been previously reported in the literature. In particular, students in the present study did not confuse the notions of average and instantaneous rates of change. Many students in the present study also conceptualize the derivative as the slope of the tangent line of its function at every point. Most students defined the integral as the area under a curve.

4. How do students view the relationship between derivative and integral? Many students in the present study considered the derivative and integral as inverses of each other. Many students’ concept maps included informal statements of the Fundamental Theorem of Calculus. For instance, Michelle wrote on her concept map, “If you take the derivative of the integral you are just left with the original
function." Most students seemed to have an intuitive understanding of the Fundamental Theorem of Calculus as evidenced by the students’ concept maps, discussions during the interviews, and other work.

**Implications for Future Research, Curriculum Development, and Teaching**

Some suggestions for future research have been alluded to in previous chapters as well as in other places in this chapter. These suggestions, as well as other recommendations for future research will be discussed in this section.

As mentioned previously, I intend to further test and refine the Physics Use Classification Scheme developed in the present study. Future investigations of the Physics Use Classification scheme could concentrate on the following questions: (1) Is the Physics Use Classification Scheme reliable? That is, do the categories accurately represent the manner in which students use physics to aid in their conceptualizations of calculus concepts? (2) Is the Physics Use Classification Scheme reproducible? That is, given a different set of students, would the same classification scheme emerge from a similar analysis? (3) Do students in each of the categories exhibit certain patterns of similarity? For instance, do most students of a certain category exhibit a common strength or weakness when solving calculus problems? Future research could look for similarities of students’ concept images within each of the Physics Use categories. In addition, students’ ability to work with average rate of change, derivative, and integral problems within a graphic, physical, symbolic and numeric context could be cross-analyzed with the Physics Use categories.
Another related direction for future research might be a more in-depth investigation of the Contextualizers. An interesting question about the nature of the Contextualizers' conceptualizations of calculus concepts arose during the analysis of Rob and Scott's data; were they thinking about actual, physical phenomena as they worked on problems, or were they remembering what the graphs of physical phenomena looked like. Future research could examine if differences exist in the way Contextualizers visualize physical phenomena and calculus concepts.

Other questions and issues for further investigation include, (1) A comparison study involving students who are and are not enrolled in an integrated calculus/physics class. In particular, do students in other types of calculus classes (traditional or reform) use physics in the same way as students in an integrated calculus/physics program? (2) How do students use their understanding of properties of derivatives (such as increasing means positive, decreasing means negative) and integrals to help them understand motion? (3) Testing the hypothesis that students are more apt to conceptualize average rate of change problems more concretely with respect to physics because they have experienced average rate of change phenomena in their everyday lives.

This study provides information about ways in which students use physics to help them conceptualize calculus concepts that will be helpful to calculus teachers and curriculum developers. In particular, the results suggest that students who exhibit difficulty working with abstract calculus concepts may benefit from relying on concrete, physical phenomena as a way to develop an understanding of calculus concepts. The integrated Calculus/Physics course drew on students' physics experiences to develop their understandings of calculus concepts. My belief is that students in the present study
developed richer understandings of the concepts of average rate of change, derivative, and integral because of the emphasis on connections between the calculus and physics in the integrated program. Teachers and curriculum developers could consider implementing some aspects of the integrated Calculus/Physics curriculum in their courses in order to promote the development of rich conceptualizations of calculus concepts.

Summary

The present study investigated how students use physics to inform their conceptualizations of calculus concepts. These uses range from using physics to interpret and visualize calculus concepts to not relying on physics knowledge to inform one’s conceptualization of calculus concepts. The major result of the present study was the development of a Physics Use Classification scheme. The Physics Use Classification Scheme is a way to categorize students based on their use of physics to solve calculus problems. The Physics Use Classification scheme was developed through a qualitative analysis of data collected from eight students throughout their enrollment in an integrated Calculus/Physics program during the 2000-2001 academic year. Each of the eight students was classified according to his/her use of physics to solve calculus problems. Directions for future investigations were presented and discussed. These suggestions include a refinement of the Physics Use Classification Scheme and further investigation of the relationship between physics use and conceptualization of calculus concepts.
APPENDIX A

STUDENT INTERVIEW PROTOCOLS
Average Rate of Change Pretest

Name: ________________________________________________

Instructions: Please answer ALL of the following questions. Show as much of your work as you can. Thank you!

1. When you hear the word “rate” what do you think of?

2. What has been your experience with the concept of rate of change (Choose one below)?

   _ I have no experience with the concept of rate of change. Please describe below what you think rate of change means.

   _ I have some informal experience with the concept of rate of change. Please briefly describe your experience below and include any definition of rate of change you can think of.

   _ I have experience with the formal definition of rate of change. Please briefly describe your experience below and include any definition of rate of change you can think of.
**PLEASE ANSWER BOTH PARTS (a) AND (b) OF QUESTION 3 & 4 – EVEN IF YOU ANSWERED ‘NO’ TO PART (a) **

3.1 Have you ever encountered the following expression or an expression similar to it

\[ \frac{f(x_1) - f(x_0)}{x_1 - x_0} \]

_Yes. If yes, when/in what context?_________________________________________________________

_No

3.2 Can you explain what the previous expression means?

4.1 Have you ever encountered the following expression or an expression similar to it

\[ f(x_1) - f(x_0) \]

_Yes. If yes, when/in what context?_________________________________________________________

_No

4.2 Can you explain what the previous expression means?
Questions 5 thru 8 refer to the following graph:

5. What is the average rate of change between $x = 1.0$ and $x = 2.0$?

6. What is the average rate of change between $x = -1.0$ and $x = 2.0$?

What is the average rate of change between $x = -3.0$ and $x = 3.0$?

What is the average rate of change between $x = a$ and $x = b$?
The position, $s$, of a car is given in the following table.

<table>
<thead>
<tr>
<th>$t$ (sec.)</th>
<th>$s$ (ft.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.0</td>
</tr>
<tr>
<td>0.2</td>
<td>0.5</td>
</tr>
<tr>
<td>0.4</td>
<td>1.8</td>
</tr>
<tr>
<td>0.6</td>
<td>3.8</td>
</tr>
<tr>
<td>0.8</td>
<td>6.5</td>
</tr>
<tr>
<td>1.0</td>
<td>9.6</td>
</tr>
</tbody>
</table>

1.1 Find the average velocity over the interval $0 \leq t \leq 0.2$.

9.2 Find the average velocity over the interval $0.4 \leq t \leq 0.8$.  

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Student Interview Protocols

Interview #1 (Average Rate of Change)

1. Think-Aloud Protocol Training
   a) Pretend it is the day that we have to turn our clocks back an hour. Describe for me, in as much detail as you can, how you would turn your clocks back an hour.
   b) Solve the following three-digit by two-digit multiplication problem (random problems were given to the students) and explain to me what you’re thinking and doing as you solve the problem.

2. Discussion of Rate of Change Pretest

3. Average Rate of Change Problem 10: The following is a graph of position versus time of an object. Find the average velocity between $t = 1$ and $t = 3$.

![Graph of position versus time]

Is it possible to tell from the graph where the velocity is positive/negative? How?
Interview #2 (Derivative)

1. Derivative Task 1: The following is a graph of a function, $f(t)$. Sketch the graph of the derivative of $f(t)$.

$f(t)$

$f'(t)$
2. Derivative Task 2: The following is a graph of a function, \( f(t) \). Sketch a graph of the derivative of \( f(t) \).
3. Derivative Task 3: The following is a graph of $g'(x)$. Sketch a graph of $g(x)$. 

![Graph of g'(x)](image-url)
4. Derivative Task 4: The following is a graph of $f'(t)$. Sketch a graph of $f(t)$ and $f''(t)$.

$f(t)$

$f'(t)$

$f''(t)$
Interview #3 (Integral)

1. Let \( F(x) = \int_0^x f(t) \, dt \) where \( f(t) \) is the function shown below.

![Graph of f](image)

(a) Does \( F(x) \) have any maximum points in the interval \([0, 10]\)?

(b) Does \( F(x) \) have any minimum points in the interval \([0, 10]\)?

2. Find the antiderivative of \( 3x^2 \).

Follow Up: When would you want to take an anti-derivative such as this?

3. Compute \( \int_0^5 3x^2 \, dx \)

Follow Up: What is the difference between questions 2 and 3?

4. If \( F(x) = \int_0^x t^4 - 2t^2 + 1 \, dt \), what is \( F(3) \)? For what values of \( x \) is \( F(x) \) positive?

5. If \( G(x) = \int_0^4 t^2 + 9 \, dt \), then what is \( G'(x) \)? What is \( G(1) \)?
Interview #4

1. The table below represents approximate values for a function, \( f(x) \) for \( 0 \leq x \leq 1 \). What can you tell me about the derivative of \( f(x) \) from the table of values?

<table>
<thead>
<tr>
<th>( x )</th>
<th>0</th>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
<th>0.7</th>
<th>0.8</th>
<th>0.9</th>
<th>1.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td>3.05</td>
<td>3.21</td>
<td>3.30</td>
<td>3.31</td>
<td>3.37</td>
<td>3.34</td>
<td>3.29</td>
<td>3.27</td>
<td>3.24</td>
<td>3.25</td>
<td>3.40</td>
</tr>
</tbody>
</table>

a) If \( g(x) = \int_0^x f(t) \, dt \), what is \( g(0.7) \)?

b) If \( h(x) = \int_{0.4}^x f(t) \, dt \), then is \( g(x) \) greater than, less than, or equal to \( h(x) \)?

2. If a spring is hanging vertically from a pole with a 20kg weight on the end and you pull slightly on the weight and then let go, what would the position, velocity, and acceleration plots of the weight look like?

3. Determine how the gravitational force between two bodies changes with respect to time if they are moving apart at a constant rate.

4. A baseball diamond is a square 90ft. on a side. A runner travels from home plate to first base at 20ft/sec. How fast is the runner’s distance from second base changing when the runner is halfway to first base? (OZ)

5. Explain what the expression \( \dot{x} = x \) if \( x(0) = 1 \) means to you.

6. Explain what the expression \( \ddot{x} = -Cx \) means to you.
APPENDIX B

2000-2001 CALCULUS/PHYSICS TOPICS SCHEDULES
<table>
<thead>
<tr>
<th>Date</th>
<th>Topic</th>
</tr>
</thead>
<tbody>
<tr>
<td>4 Sep</td>
<td>First Day of classes</td>
</tr>
<tr>
<td>5 Sep</td>
<td>Intro. to Velocity; Ch.1, 2.1-2.3</td>
</tr>
<tr>
<td>6 Sep</td>
<td>Average Velocity; Average Rate of Change</td>
</tr>
<tr>
<td>7 Sep</td>
<td>Sequences Convergence</td>
</tr>
<tr>
<td>11 Sep</td>
<td>Average Rate of Change Instantaneous Rate of Change</td>
</tr>
<tr>
<td>12 Sep</td>
<td>Vector Addition &amp; Problems; 2.4, 2.7, 2.8</td>
</tr>
<tr>
<td>13 Sep</td>
<td>Derivatives of Polynomials</td>
</tr>
<tr>
<td>14 Sep</td>
<td>Motion on inclined plane; Subtracting vectors; 3.1-3.5</td>
</tr>
<tr>
<td>18 Sep</td>
<td>Projectile Motion Video Point/demo; 4.1-4.4</td>
</tr>
<tr>
<td>19 Sep</td>
<td>Anti-Dervatives</td>
</tr>
<tr>
<td>20 Sep</td>
<td>2-dim tutorial; 4.5-4.6</td>
</tr>
<tr>
<td>21 Sep</td>
<td>Shifting Rational Functions</td>
</tr>
<tr>
<td>22 Sep</td>
<td>Problem Solving</td>
</tr>
<tr>
<td>23 Sep</td>
<td>x.u.a plots tutorial/problems; 2.5</td>
</tr>
<tr>
<td>24 Sep</td>
<td>Kinematics (graphs)</td>
</tr>
<tr>
<td>25 Sep</td>
<td>2-dim tutorial; 4.5-4.6</td>
</tr>
<tr>
<td>26 Sep</td>
<td>Review</td>
</tr>
<tr>
<td>27 Sep</td>
<td>Review</td>
</tr>
<tr>
<td>28 Sep</td>
<td>Review</td>
</tr>
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<td>29 Sep</td>
<td>Review</td>
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<td>30 Sep</td>
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<td>31 Sep</td>
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<tr>
<td>1 Oct</td>
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<td>3 Oct</td>
<td>Review</td>
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<tr>
<td>4 Oct</td>
<td>Review</td>
</tr>
<tr>
<td>5 Oct</td>
<td>Review</td>
</tr>
<tr>
<td>6 Oct</td>
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</tr>
<tr>
<td>7 Oct</td>
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<tr>
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<td>Review</td>
</tr>
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<td>Review</td>
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<td>15 Oct</td>
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APPENDIX C

STUDENT SURVEY
1. What classes, other than Calculus/Physics did you take last [Fall 2000] semester?

2. What classes are you taking this [Spring 2001] semester?

3. What clubs/sports/organizations are you involved in?

4. What was your major ENTERING the University of New Hampshire?

5. Have you changed your major? Do you plan on changing your major? Why? If you’re not planning on changing your major, why/how did you choose your current major?

6. What are your hobbies? What do you enjoy doing in your free time?

7. Why did you decide to enroll in Calculus/Physics?

8. Do you have a part-time job? If so, describe your duties. How much time per week (on average) do you work?

9. Talk about any short term or long term plans you have related to your major (i.e., getting a summer internship; working in a lab next year, etc.).

10. Please add any other comments that you would like me to know about you as a student or any non-academic interests that I haven’t asked about.
APPENDIX D

D - 1 ROB'S RATE OF CHANGE CONCEPT MAP
D - 2 SCOTT'S RATE OF CHANGE CONCEPT MAP
D - 3 TERRY'S RATE OF CHANGE CONCEPT MAP
D - 4 TODD'S RATE OF CHANGE CONCEPT MAP
D - 5 TRAVIS'S RATE OF CHANGE CONCEPT MAP
D - 6 MICHELLE'S RATE OF CHANGE CONCEPT MAP
D - 7 PAUL'S RATE OF CHANGE CONCEPT MAP
Figure D1: Rob’s Rate of Change Concept Map

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Figure D2: Scott’s Rate of Change Concept Map
Figure D3: Terry's Rate of Change Concept Map
Figure D4: Todd's Rate of Change Concept Map
Figure D5: Travis's Rate of Change Concept Map
Figure D6: Michelle’s Rate of Change Concept Map

The concept map illustrates various concepts related to rates of change, including:

- Limits
- Instantaneous rates of change
- Derivatives
- Average rates of change
- Slope

These concepts are interconnected to show the relationships and implications of each concept in the context of rates of change.
Figure D7: Paul's Rate of Change Concept Map
APPENDIX E

E - 1a ROB'S DERIVATIVE CONCEPT MAP
E - 1b ROB'S INTEGRAL CONCEPT MAP
E - 2 SCOTT'S DERIVATIVE AND INTEGRAL CONCEPT MAPS
E - 3 TERRY'S DERIVATIVE AND INTEGRAL CONCEPT MAPS
E - 4 TODD'S DERIVATIVE AND INTEGRAL CONCEPT MAPS
E - 5a TRAVIS'S DERIVATIVE CONCEPT MAP
E - 5b TRAVIS'S INTEGRAL CONCEPT MAP
E - 6 MICHELLE'S DERIVATIVE AND INTEGRAL CONCEPT MAPS
E - 7a PAUL'S DERIVATIVE CONCEPT MAP
E - 7b PAUL'S INTEGRAL CONCEPT MAP
E - 8 JASON'S DERIVATIVE AND INTEGRAL CONCEPT MAPS
Figure E1a: Rob’s Derivative Concept Map
Figure Elb: Rob’s Integral Concept Map
Figure E2: Scott's Derivative and Integral Concept Map
Figure E3: Terry's Derivative and Integral Concept Map
Figure E4: Todd's Derivative and Integral Concept Map
Figure E5a: Travis's Derivative Concept Map
Figure E5b: Travis’s Integral Concept Map
Figure E6: Michelle’s Derivative and Integral Concept Map
Figure E7b: Paul's Integral Concept Map
Figure E8: Jason’s Derivative and Integral Concept Map
APPENDIX F

INSTITUTIONAL REVIEW BOARD APPROVAL
The Institutional Review Board for the Protection of Human Subjects in Research has reviewed the protocol for your project as Exempt as described in Federal Regulations 45 CFR 46, Subsection 46.101 (b) (2), category 1.

Approval is granted to conduct the project as described in your protocol. Changes in your protocol must be submitted to the IRB for review and approval prior to their implementation.

The protection of human subjects in your study is an ongoing process for which you hold primary responsibility. In receiving IRB approval for your protocol, you agree to conduct the project in accordance with the ethical principles and guidelines for the protection of human subjects in research, as described in the Belmont Report. The full text of the Belmont Report is available on the OSR information server at http://www.unh.edu/osr/compliance/belmont.html and by request from the Office of Sponsored Research.

There is no obligation for you to provide a report to the IRB upon project completion unless you experience any unusual or unanticipated results with regard to the participation of human subjects. Please report such events to this office promptly as they occur.

If you have questions or concerns about your project or this approval, please feel free to contact me directly at 862-2003. Please refer to the IRB # above in all correspondence related to this project. The IRB wishes you success with your research.

For the IRB,

Kathryn B. Cataneo
Executive Director
Office of Sponsored Research

File
Combined Calculus and Physics Course

Kelly Black  Karen Marrongelle  Dawn Meredith
Assistant Professor  Graduate Student  Associate Professor
Department of Mathematics  Department of Mathematics  Physics Department

1. Introduction. The biggest difficulty for first-year engineering students to overcome is adjusting to the difficult academic load. The primary hurdle is the combination of the calculus and the physics courses. These courses have been offered as two disjoint topics, and many students look upon these courses as a hurdle to overcome rather than as important topics with respect to their curriculum.

The proposed pilot project is designed to combine these two important classes and make them more relevant to the students. The interplay between the two courses will allow us to examine the calculus topics within a specific context and allow us to examine the physics topics in a more meaningful way.

The pilot project will take place over a two year period. In each year, a small number of students (24) will take part in the combined courses. Students will be asked to take a pre-test when the class begins, and predetermined parts of their work will be examined throughout the semester. The work that is examined will be used to make comparisons with data sets that have already been collected at other universities.

2. Specific Aims. The ultimate goal is to augment and improve upon both the physics and calculus courses. Students will be expected to gain a deeper understanding of basic concepts. Furthermore, students will be expected to have more experiences in problem solving. That is, we expect students to have a better understanding of the problem solving process. Students will be expected to take part in the full range of activities that lead to the successful solution of a particular problem:

- read a problem statement,
- deconstruct a problem statement and decide what is being asked, what is important, and what is not important,
- decide on a course of action,
- successfully carry through the necessary steps,
- check their solution and decide if the final solution is correct.

The pilot program will also allow us to determine some of the structural difficulties in designing the course:

- what are the scheduling difficulties?
- how much material will students be able to synthesize?
- what materials will be required in addition to the books?
- what material from each of the two courses is truly disjoint?


- Setting: Before implementing the program on a larger scale, a pilot program will be initiated that will allow us to evaluate the effectiveness of the approach. Students will cover much of the same material that they would ordinarily see in
the separate courses. The principal difference will be in the order that material is covered and the classroom methodologies employed.

- **Protocols:** The students who choose to participate in the classes will be asked to take a pre-test during the first day of class. The pre-test will provide a benchmark that will allow us to determine the students general scientific background. We will also ask for each student’s SAT scores. As the course progresses, course material will be used to gain a sense of the student’s development. In the long run, we will keep track of student’s grades in the courses that require calculus and physics as prerequisite courses.

The material will be chosen from regularly scheduled homework and test problems. The material will also include pre-determined problems that will augment the homeworks and test problems. Individual students may be asked to examine sample problems that will help us determine the extent of student’s understanding of well defined, first principles. The format of these sessions may include written or oral work including “think-aloud” problems.

4. **Interpretation of Data.** Any recorded data will consist of the student’s written work or from video recordings of one-on-one sessions between the instructor and a student. We have enlisted the aid of a specialist in physics education, Randall Harrington at the University of Maine (Orono campus), to help design, implement, and analyze the materials that will be collected. The materials will be constructed during the summer of 1998.

5. **Risks.** The program will be employed on a small scale and is designed to allow informal interaction between faculty and students. All faculty involved with the project will be asked to respect the anonymity of the students.

6. **Benefits.** The program is a promising attempt to utilize calculus reform methodologies on a large scale. The pilot program will allow us to investigate whether or not the method can be implemented and allow us to investigate how well the method can be scaled to accommodate the large calculus sections at UNH.

7. **Appendices.** A Copy of the Informed Consent Form and a letter to be distributed to students are attached.


