Theoretical mechanisms for solar eruptions

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Theoretical mechanisms for solar eruptions

Abstract
This thesis presents new theoretical models of solar eruptions which are derived from older models that involve a loss of equilibrium of the Sun's coronal magnetic field. These models consist of a magnetic flux rope nested within an arcade of magnetic loop. Prior to an eruption, the flux rope floats in the corona under a balance between magnetic compression and tension forces. When an eruption occurs, the magnetic compression exceeds the magnetic tension and causes the flux rope to be thrown outwards, away from the Sun. Three important factors which impact the occurrence and evolution of the eruptive processes are investigated. These factors are magnetic reconnection, new emerging flux, and the large scale curvature of the flux rope.

First, our new results confirm that in the absence of reconnection, magnetic tension in two-dimensional configuration is always strong enough to prevent escape of the flux rope to infinity after it erupts. However, only a relatively small reconnection rate is needed to allow the flux rope to escape to infinity. Specifically, for a coronal density model that decreases exponentially with height we find that average Alfven Mach number $MA$ for the inflow into the reconnection site can be as small as $MA = 0.005$ and still be fast enough to give a plausible eruption. The best fit to observations is obtained by assuming an inflow rate on the order of $MA \approx 0.1$.

Second, we have found that the emergence of new flux system in the vicinity of a preexisting flux rope can cause a loss of ideal-MHD equilibrium under certain circumstances. But the circumstances which lead to eruption are much richer and more complicated than commonly described in the existing literatures. Our model results suggest that the actual circumstances leading to an eruption are sensitive, not only to the polarity of the emerging region, but to several other parameters, such as its strength, distance, and area as well. The results also indicate that in general there is no simple, universal relation between the orientation of the emerging flux and the likelihood of an eruption.

Finally, our research shows that the large-scale curvature of a flux rope increases the magnetic compression and helps propel it outwards. We also find that the maximum total magnetic energy which can be stored in our model before equilibrium is lost is 1.53 times the energy of the potential field, which is consistent with the theoretical limit, 1.662, for the fully opened field predicted by Aly [1991] and Sturrock [1991].

Keywords
Physics, Astronomy and Astrophysics, Physics, Electricity and Magnetism

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THEORETICAL MECHANISMS FOR SOLAR ERUPTIONS

BY

Jun Lin
B. S., Nanjing University, China, 1985
M. S., Nanjing University, China, 1988

DISSERTATION

Submitted to the University of New Hampshire
in partial fulfillment of
the requirements for the degree of

Doctor of Philosophy
in
Physics

September 2001
This dissertation has been examined and approved.

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Senior Scientist of Solar Physics, Harvard-Smithsonian Center for Astrophysics

May 16, 2001
Date
There is nothing new under the Sun, but there are lots of old things we don't know.

—Ambrose Bierce
Dedication

To my grandma, my parents, and my sister

my wife, Nina

and my son, Mingkai
Acknowledgments

My studies and research work for the PhD program here at UNH were supported by NASA grants NAG5-4856, NAG5-1479, NAG5-8228, NSF grant ATM-9808063, and Lockheed-Martin grant NAS8-37334 to the University of New Hampshire.

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Dr. Gu helped me again in 1994 to obtain independent funding from K. C. Wong Education Foundation of Hong Kong through the aegis of the Chinese Academy of Sciences. This fund supported a one-year visit to the University of New Hampshire, to continue to work with Professor Forbes as a research scholar. It was following this visit that Professor Forbes applied for funding from NASA to make it possible for me to read for my PhD in Physics at UNH.

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ABSTRACT

THEORETICAL MECHANISMS FOR SOLAR ERUPTIONS

by

Jun Lin
University of New Hampshire, September, 2001

This thesis presents new theoretical models of solar eruptions which are derived from older models that involve a loss of equilibrium of the Sun’s coronal magnetic field. These models consist of a magnetic flux rope nested within an arcade of magnetic loop. Prior to an eruption, the flux rope floats in the corona under a balance between magnetic compression and tension forces. When an eruption occurs, the magnetic compression exceeds the magnetic tension and causes the flux rope to be thrown outwards, away from the Sun. Three important factors which impact the occurrence and evolution of the eruptive processes are investigated. These factors are magnetic reconnection, new emerging flux, and the large scale curvature of the flux rope.

First, our new results confirm that in the absence of reconnection, magnetic tension in two-dimensional configuration is always strong enough to prevent escape of the flux rope to infinity after it erupts. However, only a relatively small reconnection rate is needed to allow the flux rope to escape to infinity. Specifically, for a coronal density model that decreases exponentially with height we find that average Alfvén Mach number $M_A$ for the inflow into the reconnection site can be as small as $M_A = 0.005$ and still be fast enough to give a plausible eruption. The best fit to observations is obtained by assuming an inflow rate on the order of $M_A \approx 0.1$.

Second, we have found that the emergence of new flux system in the vicinity of a pre-
existing flux rope can cause a loss of ideal-MHD equilibrium under certain circumstances. But the circumstances which lead to eruption are much richer and more complicated than commonly described in the existing literatures. Our model results suggest that the actual circumstances leading to an eruption are sensitive, not only to the polarity of the emerging region, but to several other parameters, such as its strength, distance, and area as well. The results also indicate that in general there is no simple, universal relation between the orientation of the emerging flux and the likelihood of an eruption.

Finally, our research shows that the large-scale curvature of a flux rope increases the magnetic compression and helps propel it outwards. We also find that the maximum total magnetic energy which can be stored in our model before equilibrium is lost is 1.53 times the energy of the potential field, which is consistent with the theoretical limit, 1.662, for the fully opened field predicted by Aly [1991] and Sturrock [1991].
Chapter 1

Introduction

The Sun, our star, is the most important object in the heavens for us. It sustains almost all of the life on the Earth and impacts many aspects of our daily life. Also as our nearest star, the Sun offers a unique opportunity to study stellar physics in action.

An eruptive solar flare is the most violent energy release process which occurs in the solar system. A major eruption usually releases more than $10^{32}$ ergs of energy and ejects more than $10^{16}$ grams of mass into planetary space. Some of this mass in the form of energetic particles with energy in the range of 10 keV to 1 GeV can seriously damage satellites as well as disrupt ground communications and power grids. Thus, studying solar eruptions and understanding the physics behind them are of broad scientific and socio-economic significance.

Generally, the radiative power output of the Sun is quite stable at $3.86 \times 10^{33}$ erg/s. The power fluctuation due to solar activity is very tiny compared to the total radiative output, constituting less than $10^{-4}$ of the total output. However, much of the output due to solar activity occurs as highly energetic radiation, so that if one looked at the Sun in UV or X-ray, fluctuations $> 10^5$ are seen.

During a large eruption a huge amount of energy (up to $10^{32}$ ergs) is released. As a result, the equivalent of about 1500 Gigawatts of electricity, which doubles the power generating capacity of the entire United States, can be flowing into the Earth’s atmosphere and near space environments and wreak havoc on a world that has come to depend on satellites,
Figure 1-1: Composite image of the Sun (at left) showing a solar eruption in progress. At right is an artist's sketch of the terrestrial magnetosphere being impacted by solar ejecta. Distances are not to scale.

electrical power, and radio communications (Figure 1-1). For instance, a series of flares and CMEs in March 1989 bombarded 1500 satellites with energetic particles and X-rays for many hours. Worse still, this bombardment led to a heating of the Earth atmosphere, causing it to expand to such an extent that many of the satellites slowed down and dropped several kilometers in their orbit due to the increased atmospheric drag. The solar panels on these satellites were degraded, and many satellites had their on board computers severely damaged. In 1994, two Canadian satellites were shut down because of radiation damage to their electronics, and as a result, telephone service across Canada was disrupted for months¹.

Misfortune can also strike scientific satellites. Japan's Advanced Satellite for Cosmology and Astrophysics (ASCA) was badly damaged by an eruption (specifically, a flare-associated

¹Find the details at: http://umbra.gsfc.nasa.gov/solar.connections/relevance.html
coronal mass ejection) on July 14, 2000. This was an unusually large event (the kind that occurs about once in 10 years) with an extremely strong solar proton flux. Due to this event, the Earth’s atmosphere rapidly expanded, and the atmospheric gas density at the ASCA altitude suddenly increased to several times its normal value. The orbit of ASCA was thus altered, the battery cells were unrecoverably damaged, and the satellite was forced to stop its normal observations².

High energy particles and emissions from solar eruptions can also change the altitude of Earth’s ionosphere and interfere with radio signals. In March 1989, listeners in Minnesota reported that they could not hear their local radio station, but that they could hear broadcasts of the California Highway Patrol (more than 2000 km away). In extreme cases, solar eruptions can completely wipe out radio communication around the Earth’s North and South Poles for hours up to days.

Solar eruptions can generate magnetic storms within the Earth’s magnetosphere that affect the strength of Earth’s magnetic field. Changes in magnetic fields can produce inductive surges in power lines and strong electric currents in gas and pipelines that can cause pipelines to corrode and deteriorate faster than they would naturally. In power lines, the extra induced current can burn out transformers leading to brownouts and blackouts. During the March 1989 storm, a transformer burned up at a power plant in New Jersey, and a whole substation of transformers was blown out at a power station in Quebec, Canada, leaving 6 million people in both Canada and the US (including New Hampshire) without electricity for hours, some for months³.

There are many kinds of disruptive processes which are often observed on the Sun, such

²More details can be found at: http://asca.gsfc.nasa.gov/docs/asca/safemode.html

³Find the details at: http://www.mpelectric.com/storms

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as surges, X-ray jets, sprays, flares, prominence disruptions, and coronal mass ejections (CMEs). A surge is a small scale (< 10^4 km), low speed ejection of cool (∼ 10^4 K) chromospheric material. They typically originate near sunspot pneumbrae [Giovanelli and McCabe 1958] or near magnetic polarity reversal lines [Rust 1968]. Their velocity varies from 100 to 200 km/s, and in most cases, they are accompanied by a small flare or chromospheric brightening. Observations show that surges generally either travel along a magnetic loop or an extended field line for a short distance before falling back to the Sun [Rust et al. 1980 and Gu et al. 1994]. Some surges show helical motions [Gu et al. 1994] indicative of twisted magnetic field lines [Canfield et al. 1996]. Surges are generally observed in the Hα and other Balmer lines of hydrogen. They have been studied for more than a half century (see Roy [1973a, b]). Another ejection that is often observed in Hα is the spray. A spray differs from a surge in that it does not fall back to the Sun in Hα movies. This implies that either the sprays do escape from the Sun or they fall back to a point when they are no longer visible in Hα. According to the motion patterns of sprays, however, the first probability is more likely to be true, because sprays are often observed at very high speed, frequently exceeding the velocity of escape. Contrasting with surges, matter in sprays appears not to be contained by the magnetic field in narrow-band observations, but to fly out in fragments [Warwick 1957; and Zirin 1966]. These features are more like those of eruptive filaments or coronal mass ejections.

Hot (10^6 K) X-rays jets were first clearly observed by the Soft X-ray Telescope (SXT) on board the Yohkoh satellite [Tsuneta et al. 1991]. They appear as thin extended features with lengths varying from 5×10^3 to more than 3×10^5 km with plasma flows in the range of 30 to 300 km/s. Almost all of them are associated with small flares in X-ray bright points, emerging flux regions, and active regions. Some jets show motion perpendicular to the
elongating direction [Shibata et al. 1992; Shibata et al. 1994; and Shimojo 1994]. X-ray jets are often accompanied by Hα surges. Canfield et al. [1996] investigated nine such events, observed simultaneously as X-ray jets and Hα surges that were associated with moving magnetic bipoles. Their work indicates that both X-ray jets and Hα surges are the high and low temperature compositions of outflow plasma most likely produced by the magnetic reconnection of oppositely directed fields forced together by the motion of the bipoles.

Both Hα surges and X-ray jets are low energy phenomena and do not have any large scale significance, nor do they have any impact on the Earth or its near space environment. Those phenomena that can impact the environment around the Earth are sprays, flares, prominence disruptions, and coronal mass ejections (CMEs). Nowadays, the spray is considered to be essentially the cool temperature part of CMEs rather than a separate phenomenon (see Rust et al. [1980]).

The solar flare is the first type of solar eruption ever to be discovered and reported in the scientific literature [Carrington 1859 and Hodgson 1859]. Since their discovery, solar flares have been studied for more than one and a half centuries. In the next chapter, we consider the flare phenomenon in more detail and discuss its role in solar eruptions.
Chapter 2

History of Research on Solar Eruptions

The first type of solar eruptive phenomenon to be observed was the flare. As we will discuss below in more detail, a flare is a sudden brightening of emission from the localized region on the surface of the Sun (the flare ribbons), and in the coronal region above them (the flare loops). With the development of new techniques, it has gradually become apparent that the surface flare is often just a secondary effect of a general disruption of the coronal magnetic field that accompanies the eruption of prominences and CMEs.

2.1 Early Years of Flare Research

As Carrington [1859] was engaged in his daily sunspot drawing in the afternoon of September 1, 1859, he noticed the sudden appearance of small bright patches in the region between the spots (Figure 2-1). After confirming that those patches were not caused by stray light, he ran to find someone to verify the observation and when he returned “within 60 seconds”, he was mortified to find that the patches had already changed a lot and were greatly enfeebled. Fortunately, his observation was independently confirmed by Hodgson [1859], an amateur astronomer who was observing at a separate site. What Carrington saw was a rare white-light flare (most flares are visible only with the use of special filters) and it was followed, some seventeen hours later, by a huge magnetic storm and brilliant aurora that could be
seen as far south as Honolulu, 21° from the magnetic equator [Kimball 1960]. Carrington [1859] hesitately suggested that the flare and the storm were related since he had no way of knowing whether their close association in time might just be a coincidence. Seventy-eight years later, Bartels [1937] established that Carrington's suggestion was correct.

Although the first solar flare observed was a white-light flare (i.e. detectable in the continuum of the solar spectrum), most flares traditionally are only visible when filters are used to block the light emitted from the photosphere in order to see the line emission that originates in the chromosphere. The most important line for flare observations is that of the \( n = 3 \rightarrow n = 2 \) transition in the Balmer series, i.e., H\( \alpha \). This is why most flare images are taken in H\( \alpha \). However, flare associated emissions actually occur over a very broad range of wavelength: from 0.002 Å for \( \gamma \)-ray emissions up to more than 10 km for radio emissions.

Before Hale [1892] invented the spectroheliograph, a device to image the Sun in H\( \alpha \), flares could be observed only in white-light. Due to the rarity of white-light flares and, more to the point, to the lack of an instrument that could efficiently image flares in the narrow emission lines in which they are most prominent, the progress in flare physics following Carrington's observations was slow.

### 2.2 Flare Observations in H\( \alpha \)

After the invention of the spectroheliograph [Hale 1929], knowledge of flares and their relationships with other eruptive phenomena accumulated rapidly. McMath et al. [1937] reported a flare associated with an erupting prominence moving outward at a speed of about 700 km/s — greater than the solar escape velocity, and visible to a distance of about \( 10^6 \) km. In other events, velocities range from 400 km/s to 1400 km/s [e.g., Dodson and McMath 1952; Reid 1959; and Valnicek 1962].
Figure 2-1: The first reported flare, on September 1, 1859, observed by Carrington in white light. The flare is the pair of crescent-shaped objects labeled A and B. The cause of white-light flare lasted for just a few minutes, and the ribbons migrated to positions C and D before fading from the view. The dark regions in the sketch are sunspots (from Švestka and Oliver [1992]).
On the solar disk, the spectrohelioscope sees flares as bright ribbons in Hα that usually appear in pairs moving away from one another at speeds that can be as large as 100 km/s, but decreases over several hours to less than 1 km/s [Dodson 1949; Dodson and Hedeman 1960; Švestka 1962; and Malville and Moreton 1963]. Flares almost always occur in regions where the surface field is complex [Waldmeier 1938; Giovanelli 1939], and the flare ribbons are always seen on opposite sides of a magnetic polarity reversal line [Bamba 1958; and Severny 1958].

Accompanying the flare ribbons is a system of flare-loops which initially appears at low altitude and then moves upward into the corona in consort with the motion of the ribbons [Moore et al. 1980]. A classical description of flare loops as seen in Hα was first given by Bruzek [1964a], who noted that the ribbons essentially lie at the footpoints of the loop system which forms an arcade of loop. Many subsequent observations in Hα, EUV, and soft X-rays have since confirmed his conclusions [Bruzek 1964b; Neupert et al. 1974; Kahler et al. 1975; Cheng and Widing 1975; Nolte et al. 1979; Martin 1979; Pallavicini and Vaiana 1980]. Doppler-shift measurements also show that the apparent motions of the loops and ribbons are not caused by mass motions of the plasma, but rather by the continual propagation of an energy source onto new field lines [Schmieder et al. 1987]. High resolution observations have also shown that the cooler loops are nested below the hotter ones with the coolest loops, which are seen in Hα, rooted at the inside edges of the ribbons [Roy 1972; and Rust and Bar 1973]. By contrast, the hottest loops, seen in X-rays, are rooted in the outer portions of the ribbons [Moore et al. 1980].
2.3 Other Observations

The development of advanced technology has opened more windows for observing the Sun, and flare research now extends to a much wider range of wavelengths than ever before.

2.3.1 Radio Bursts

In 1942, meterwave radiation from the Sun was recorded for the first time with the British army’s radar. Because it was during World War II, the discovery was kept secret and it was not until 1945 that the first well-documented example of meterwave bursts and their association with solar flares was published [Appleton and Hey 1946]. Today, we know that the metric component as well as impulsive microwave component of radio burst are identified with the flare impulsive phase, and are evidence of particle acceleration and plasma heating [Švestka 1976, Wild 1985; and Suzuki and Dulk 1985].

2.3.2 Energetic Particle Events

Some flares or flare-associated processes can produce an observable, transient increase in cosmic ray flux at the surface of the Earth because they produce protons with energies in excess of 500 MeV. Such flares are known as cosmic-ray or proton flares. Like white-light flares, they are rare. Within the thirty years from 1942, when the first cosmic-ray flare was observed [Forbush 1946], to 1972, only twenty such events were recorded.

Proton flares typically show all the features of major flares, such as two bright ribbons in chromosphere, and strong radio bursts of various types (Ellison et al. [1961]; Sawyer [1968]; Martres [1968]; for a review see Švestka [1968 and 1976]). Traditionally, the origin of the energetic particles was ascribed to some unknown process producing the flare emissions in Hα and X-rays. However, comparison of durations of solar energetic particle events
with the durations of associated flare emissions raised suspicions about a close association between the two [Meyer et al. 1956]. Reames [1992] has shown that virtually all solar energetic particles that reach the Earth are created by Fermi-type process operating at the interplanetary shock in front of the CME that always accompanies a large flare.

Besides emissions in $\text{H}_\alpha$, radio wavelengths, and energetic particles, flares also produce hard X-ray ($>10$ keV) and $\gamma$-ray emissions. What gives rise to these emissions is still unknown, but is generally thought that they are produced by a class of energetic electrons and the protons that bombard the solar surface. They do not seem to contribute significantly to the energetic particle population that reaches the Earth.

### 2.4 Observations in Space

Prior to 1960, solar flare research relied heavily on observations in $\text{H}_\alpha$. In the 1960s, our knowledge of solar flares increased dramatically with the OSO series of satellites that allowed us for the first time to study in detail the characteristics of flares at wavelengths inaccessible to ground-based observatories. These observations were soon supplemented by others, such as TD-1A in 1972, Skylab in 1973-1974, and P78-1. Of particular interest were the data from SMM and Hinotori spacecrafts, with their emphasis on the high-energy aspects of solar flares. As the techniques of photography started to be used to study solar flares in X-rays in the 1960s (the first direct photograph of solar flare in X-ray between $3$ and $14$ Å was obtained in 1968 during a rocket flight [Vaiana et al. 1968; and Vaiana and Giacconi 1969]), flare physics entered a new era of discoveries.

Although the resolution of the first X-ray photograph of a flare was poor (approx 20" compared to $\approx 1$" available today), the general correspondence of bright regions in X-ray to bright regions in $\text{H}_\alpha$ was quite apparent. Apart from the bright regions visible on the
surface in both X-ray and Hα. bright X-ray loops were found to be lying above the cooler Hα loops. This verified what had long been suspected, namely that the flare loop system includes very hot components in addition to the cool (Hα) one observed by ground based instruments.

2.5 Models for the Formation of Flare Loops and Ribbons

In the previous sections, we briefly discussed the progress in observations of solar flares. In this section, we take a backward look at the historical progress made in understanding the physics underlying the phenomena of flare loops and ribbons.

It was known for sometime that there was a continual down flow of material in the cool Hα loops throughout their existence. Early on, it was thought this could be explained as a condensation of the hot coronal plasma due to a thermal instability. However, Kleczek [1964] estimated that the total mass delivered to the chromosphere by this downflow is about $10^{16}$ g, which is almost the total mass in the whole corona, so, it is difficult to account for it by condensation of coronal material [Jefferies and Orrall 1963, 1965a and b; Kleczek 1963 and 1964]. To make matters worse, these estimations were made on the basis of Hα observations alone. Later on, Pneuman [1981] integrated the derived density in the flare loop system for the July 29, 1973 flare using X-ray observations. He obtained a total mass for that event of $7.5 \times 10^{16}$ g, and concluded that this much greater mass implied that the higher loops are never seen in Hα because they never cool below coronal temperature [Pneuman and Orrall 1986]. We will discuss this conclusion later and will find that the reason we cannot observe Hα loop systems at higher altitude is because the magnetic field lines, on which the loops are supposed to lie, undergo a shrinkage process as they are cooling from high temperature to low temperature [Švestka et al. 1987; Lin et al. 1995; and Forbes and Acton 1996].
Ruling out coronal condensation as the mass supply means that the material must somehow be supplied from the chromosphere and that a hot upflow is invisible in Hα. To account for this upflow, the process of chromospheric "evaporation" was suggested by many authors [Hudson and Ohki 1972; Sturrock 1973; Hirayama 1974; Lin and Hudson 1976; Antiochos and Sturrock 1978; Colgate 1978; and Withbroe 1978]. In this process chromospheric material is heated either by energetic particles [Sturrock 1973; and Lin and Hudson 1976] or by thermal conduction along field line from an energy source located above the flare loops [Hirayama 1974; Antiochos and Sturrock 1978; Colgate 1978; and Withbroe 1978].

Carmichael [1968] was the first to suggest that the flare loops and ribbons could be understood as a consequence of the relaxation of magnetic field lines stretched by the ejection of plasma into interplanetary space. Because the field lines of closed magnetic loops are well anchored at their footpoints in the photosphere, they become highly extended when the plasma at the top of the loop is ejected during an eruption. The field lines stretched in this way are said to be "open", and then they relax to form closed loops through a process known as magnetic reconnection (see Figure 2-2). Models of this process show that the reconnection process also releases sufficient magnetic energy to account for the radiative and kinetic energy generated during an eruption (see also Sturrock [1968]; Kopp and Pneuman [1976]; Bruzek [1969]; Sturrock [1972]; Roy [1972]; and Hirayama [1974]). The rise of the loop system is explained by the fact that the reconnection site continually moves upward as more and more magnetic field lines reconnect. This picture automatically accounts for the apparent motion of flare ribbons without the existence of any actual plasma flow in the ribbons. It also explains why the hottest X-ray loops are at the top of the loop system [Kopp and Pneuman 1976; Heyvaerts et al. 1977; Cargill and Priest 1982; and
Figure 2-2: The basic coronal magnetic field configuration first proposed for eruptive flares by Carmichael [1964] (upper left), later improved by Sturrock [1968] (upper right), Hirayama [1974] (middle), and eventually by Kopp and Pneuman [1976] (bottom). (From Švestka and Cliver [1992].)
Pneuman 1981].

However, the above picture does not explain how it is possible for mass to flow downward in the loop system during its life time without draining virtually all the mass in the corona. Forbes and Malherbe [1986a] showed that this could be explained by chromospheric evaporation acting along field lines mapping into the reconnection region as illustrated in Figure 2-3. Various numerical simulations confirming this picture have been carried out by Forbes and Malherbe [1986a and b], Forbes et al. [1989], Forbes and Malherbe [1991], and Yokoyama and Shibata [1997 and 1998].

According to Forbes and Malherbe [1986a and b], the flare loops are created by chromospheric evaporation (which they prefer to call ablation) on field lines mapping to slow-mode shocks in the vicinity of reconnection site. The shocks annihilate the magnetic field in the plasma flowing through them, and the thermal energy which is thus liberated is conducted along the field to the chromosphere. This in turn drives an upward flow of dense, heated plasma back toward the shock and compresses the lower regions of the chromosphere downward.

2.6 Coronal Mass Ejections and Eruptive Prominences

Because the study of solar flares dominated solar physics research for a long time, we have given it priority in this historical review. Although flares are still an important topic, they no longer dominate research on solar eruptions that effect the Earth and its near space environment. In this section, we are focusing on another aspect of solar eruptions, which is now dominant, namely the coronal mass ejection (CME). CMEs are closely associated with the eruption of solar prominences.
Figure 2-3: Schematic diagram of a two-ribbon flare. Solid lines indicate boundaries between various plasma regions, while dashed lines indicate magnetic field lines. Because of the assumed symmetry, only left part of the configuration is drawn (from Forbes and Acton [1996]).
2.6.1 Prominences

Prominences were observed long before flares. They can easily be seen with the naked eyes during a total eclipse of the Sun. So, they may well have been observed by our early ancestors. The earliest scientific record of a prominence can be traced back to 1239 when Muratori observed the corona during a total eclipse and reported a "burning hole" in it [Secchi 1875]. At this early time, many people believed prominences to be clouds in the lunar, rather than solar, atmosphere. Later, once it became possible to see prominences without an eclipse, they were thought by some to be the mountains on the Sun [Grant 1852]. Not until the 1860's, when Secchi [1875] introduced photography and spectrographic methods [Secchi 1868; de La Rue 1868], did it become clear that prominences are glowing masses of gas.

With the invention of the spectroheliograph and the spectrohelioscope, routine observations of prominences became possible. Prominences appear as absorption features when seen on the solar disk and as emission features when seen on the limb. Both Hale and Ellerman [1903], and later Deslandres [1910], realized that the absorption features, generally referred to as filaments, on the disk in their spectroheliograms, are nothing more than prominences seen against the bright photosphere. Later, with the invention of the coronagraph [Lyot 1939], another milestone in the development of solar physics after Hale's inventions, it became possible routinely to observe prominences on the limb at any time without waiting for the next total eclipse. For more on prominences, the readers should consult Tandberg-Hanssen [1995]).

Today, the term prominence is primarily used to refer to large quiescent prominences, but it also includes the smaller ones that often occur in active regions (i.e. regions containing sunspots). Prominences consist of cool ($\approx 10^4$ K) partially ionized plasma which is more
than a factor of 10 denser than the hot ($> 10^6$ K) coronal plasma in which they are suspended. They are supported against gravity by a nearly horizontal magnetic field whose exact geometry is still uncertain. Quiescent prominences can last for more than three months as relatively stable structures, but eventually they erupt outwards. Although some prominence material falls back to the Sun, most of it is ejected into interplanetary space.

The life-time of a prominence in active regions is a few days. Observations show that a prominence eruption, whether inside or outside an active region, is always accompanied by a CME. Active prominence eruptions are also nearly always accompanied by a major flare, but quiescent prominence eruptions produce only weak chromospheric brightening which is too weak to be considered a flare. The reason for the difference in the brightness of the emission is almost certainly due to the higher strength of magnetic field in the active region (almost a factor of 100). More details of observations are referred to Zirin [1988] and Tandberg-Hanssen [1995]. A summary of theoretical models of quiescent prominences was presented by Priest [1982].

### 2.6.2 Coronal Mass Ejections

Coronal mass ejections (CMEs) are commonly defined as large-scale ejections of mass and magnetic flux from the lower corona into interplanetary space. The term CME originally referred to the observation of material being ejected from the Sun and traveling through the field of view of a white light coronagraph. Measurements from coronagraphs and spacecraft show that a typical CME injects roughly $10^{23}$ Mx of magnetic flux and $10^{16}$ g of plasma into space [Howard et al. 1985; Hundhausen 1988 and 1999; Gosling 1990; Webb et al. 1994], and that the total kinetic energy involved in such an ejection can reach up to $10^{32}$ erg - equivalent to that released during a major flare. Individual CMEs cause significant
disturbances in the solar wind, and they are the main driver of "space weather" at Earth orbit [Fox et al. 1998]. Their contribution to the average solar mass loss rate is small, constituting only about 10% of the mass loss rate due to the steady solar wind [MacQueen 1980; Webb and Howard 1994]. The detailed structure of CMEs varies from event to event, but the basic structure appears to be nearly the same in all events, consisting of three main components when observed in white light [Fisher and Poland 1981; Low et al. 1982; Illing and Hundhausen 1985 and 1986; Howard et al. 1985; Hundhausen 1988 and 1999; and Dere et al. 1999]. They have a bright and high-density front (or leading arc) moving ahead of a dark and low-density cavity within which a bright and relatively high-density core is found. Many CMEs show low-density cavities without a bright core. These cavities sometimes contain no high-density core, either because such a core is really absent or perhaps because the Thompson-scattered light from the core cannot be seen from some perspectives [Gibson and Low 1998].

Bright cores were found in about one-fourth of the CMEs observed by the Solar Maximum Mission (SMM) satellite, and the bright leading arc generally moves in radial direction at speed from less than 50 km/s to more than 2000 km/s [Howard et al. 1985; Hundhausen et al. 1994]. After onset, CMEs usually undergo continuous, or discontinuous, accelerations. For the so-called impulsive high speed CMEs, deceleration is also observed [Sheeley et al. 1999]. Interplanetary shocks often form in front of CMEs and the strength of shock depends on the configuration of interplanetary magnetic field [Gosling 1993]. During the quiet phase of the solar cycle there are approximately two CMEs per week, but during the active phase the rate can exceed one per day. The sensitive observations made by the LASCO (Large Angle Spectrometric Coronagraph) on board the SOHO satellite yielded a CME rate in 1996 that was a factor of two to three higher than during the previous solar
Coronal mass ejections were known as coronal transients before space observations. According to Rust et al. [1980], coronal transients were mass ejections seen in the inner corona in white-light by ground based coronagraphs whose radial field of view is too limited to clearly see the three part structure of most CMEs. As we mentioned before, Carrington [1859] suggested the possibility that geomagnetic disturbances were caused by solar eruptions. However, it was Lindemann [1919] who suggested for the first time that geomagnetic storms result from transient ejections of plasma from the Sun. It was only realized much later that such transient ejections of solar material should drag magnetic loops out into interplanetary space (e.g., Cocconi et al. [1958]; Piddington [1958]; Gold [1962]), because of the high electrical conductivity of the coronal plasma. Gold [1955] was the first to suggest that the transient produce a shock disturbance in the interplanetary gas as they propagate from the Sun — an idea which is based on the extremely rapid onset of the magnetic disturbance at the Earth (the so-called “sudden commencement”). It was widely believed at that time that these transient plasma ejections were a consequence of solar flares [Hale 1931; Chapman 1950; Piddington 1958].

White-light coronal transients (mass ejections) were first observed in space by the NRL coronagraph on OSO-7 [Tousey 1973], but it was still assumed they were directly generated by flare activity. However, with time it eventually became clear that only about 10% of all flares (mostly the very large flares) generate CMEs. A far better correlation was found between CMEs and prominence eruptions.

There is now a wide variety of observations that indicate that CMEs are not, in general, produced directly by impulsive solar flares [Gosling 1993]. UV and X-ray telescopes on SOHO and Yohkoh have found that CMEs often begin to lift off from the Sun before any
substantial flaring activity occurs [Wagner et al. 1981; Harrison 1986 and 1995; Hundhausen 1988; Harrison et al. 1990; Hundhausen 1997]. Furthermore, the thermal energy produced by the flares is not sufficient to propel them at the high speeds (up to 2000 km/s) that are often observed [Canfield et al. 1980; Webb et al. 1980; Howard et al. 1985; Linker et al. 1990; and Hundhausen et al. 1994].

Statistical association studies indicate a higher correlation between CMEs and erupting prominences than with other activity [St. Cyr and Webb 1991; and Dryer 1994]. According to Webb [1992], 88% of CMEs are associated with erupting prominences, while only 34% are associated with Hα flares (this percentage added up to > 100% because 12% of prominence eruptions are accompanied by a flare). In fact, erupting filaments and X-ray events, especially those of long duration, are the most common near-surface activity associated with CMEs. Comparisons of soft X-ray data with the white light observations have given us many insights into the source regions of CMEs. Using whole-Sun integrated X-ray data, Sheeley et al. [1983] first showed that the probability of associating a CME with a soft X-ray flare increased linearly with flare duration, reaching 100% for flares of duration ≥ 6 hours.

Observations show that the events associated with CMEs generally have a gradual phase lasting several hours. Some events have γ-ray emissions that last for as long as 8 hours [Kanbach et al. 1993; and Ryan 2000] and X-ray emissions lasting as long as 50 hours have been observed in some major events. The emission is generally produced by energetic charged particles that are related to the process of magnetic energy release. Magnetic reconnection is commonly thought to be the likely mechanism accelerating the particles that create these emissions. The acceleration may occur directly because of the electric field associated with the reconnection process or indirectly because of a stochastic process.
resulting from turbulence in the outflow from the reconnection site [Ryan and Lee 1991]. However, the energetic particles that reach the Earth are unlikely to be produced by the reconnection process. They are almost certainly produced at the fast shock in front of the CME by diffusive shock acceleration (Reames [1995]), and it is these particles that give rise to solar energetic particles, namely SEPs (Kahler [1994]. See also the studies by Reames [1995 and 1997], and Gosling [1997]).

2.7 Summary

As more observations have become available, the boundaries separating the flare, erupting prominence and CME phenomena have become increasingly blurred. It seems likely that as our knowledge of these events increases, we will find that there are manifestations of a single physical process involving the disruption of the coronal magnetic field.
Chapter 3

Review on the Theory of Solar Eruptions

3.1 Early Models

Following observations of CMEs in the 1970s, many authors immediately tried to construct the theoretical reasons for this important and complex eruptive phenomenon, and its relations to flares and filament eruptions. In early theoretical models, the observed association between CMEs and flares naturally suggests the possibility, considered in the late 70s, that CMEs are the dynamical response of the corona to the sudden input of energy liberated by a flare at the base of the corona [Dryer 1982]. Some of these models were almost purely hydrodynamical [Wu et al. 1975] and were based on numerical simulation of the response of a model atmosphere to an ad hoc pressure or velocity pulse at the base. These models were extended to include two-dimensional magnetohydrodynamic (MHD) processes later on [Nakagawa et al. 1978 and 1981; Wu et al. 1978 and 1981; Steinolfson et al. 1978; and Dryer et al. 1979]. In these models, magnetic forces had only a passive role as an inhibitor or guider of the transient material. These models suffer the general problems of requiring unrealistically low magnetic field strength (relative to the gas or ram pressure) or initially opened field topologies rather than the closed fields observed.

Realizing that the corona is dominated by the magnetic field, some researchers investigated the roles played by the magnetic field in triggering CMEs. Sakurai [1976], Anzer
and Mouschovias and Poland [1978] proposed models whereby a twisted flux-tube is driven outward by its stored energy. Pneuman [1980a] developed a model based on the close association between CMEs and eruptive prominences. He showed that an increase in the strength of the magnetic field beneath a coronal helmet streamer can easily propel the prominence and the overlying arcade outward to infinity. Pneuman [1980b] and Anzer and Pneuman [1982] argued that in two-ribbon flares this driving force can originate via the outflow from reconnection beneath the rising prominence. Other possibilities of triggering CMEs were also considered based on some kind of magnetic driving [Liu 1983; Yeh and Dryer 1981; Yeh 1982 and 1983; Hu and Tang 1984; and Hu and Jin 1987]. Although they could reproduce some observational features of CMEs, these models were still based on the principle that the driving forces somehow originated in the flare.

### 3.2 Storage Models

The most generally accepted models now assume that the energy released during eruptions is stored in the coronal magnetic field prior to the eruption, and that a loss of stability or equilibrium of the coronal magnetic field leads to the eruption and energy release.

These models transfer energy from the convection zone over a long time scale. The continual emergence of new flux from the convection zone and the movement of the footpoints of closed coronal field lines causes stresses to build up in the coronal field. Eventually, these stresses exceed a threshold beyond which a stable equilibrium cannot be maintained, and the field erupts. These models are based on the mechanism of accumulating energy in the coronal magnetic field, so they are thought of as storage models [Priest and Forbes 2000].

In these models, energy accumulation generally takes tens of hours or even a couple of days. During this process, the coronal magnetic field evolves from a potential field to a non-
potential field, and the magnetic free energy increases. The free energy is the difference of the energy between non-potential field and potential field. As the energy stored in the system increases, the configuration may become unstable or reach the point where no equilibrium is possible, leading to a so-called "loss of equilibrium". In the next section, we discuss two specific storage models, which have been extensively studied, namely the sheared arcade model and the flux rope model.

Based on many observations, an important constraint for CME and related eruption models is that the normal component of the photospheric magnetic field remains virtually unchanged during the event. The slow movements of sunspots and other magnetic features in the photosphere are unaffected by the eruptions because the plasma in the photosphere is almost $10^9$ times denser than the plasma in the corona where eruptions originate. This enormous difference in density, and thus in inertia, means that it is very difficult for disturbances in the tenuous corona to have much effect on the extremely massive plasma of the photospheric layer. Field lines mapping from the corona to the photosphere are said to be "inertially-tied" or "line-tied", which means that the footpoints of coronal field lines are essentially stationary over the time scale of the eruption. Therefore the component of the coronal field due to photospheric sources remains constant during an eruption and does not contribute to the energy release.

### 3.3 Energetics

Forbes [2000] estimated that the energy density required to drive a moderately large eruption of $10^{32}$ ergs (i.e. a flare associated with a CME) must be of order 100 ergs/cm$^3$. Comparisons of possible energy sources in the corona show that only the magnetic energy density exceeds this amount. The magnetic energy must be accumulated in the components of the magnetic
field generated by coronal currents built up either by the observed photospheric motions or transported into the coronal by flux emergence. The relatively short time interval between large events implies that much of the current is transported from below [McClymont and Fisher 1989]. Because there is no way to observe the coronal magnetic field, its exact form remains unknown. It is most commonly assumed that the field is in the form of magnetic arcades or magnetic flux ropes with the current flowing parallel to the field, known as force-free configurations since $\mathbf{J} \times \mathbf{B} = 0$ [Van Ballegooijen and Martens 1989; Ridgway and Priest 1993; and Van Ballegooijen 1999].

Related to the question “How much energy is required to drive a CME?”, another question is “Can the magnetic configuration store enough energy before eruption?”. Low and Smith [1993] pointed out that the amount of energy should be enough to do three things: (1) open up the magnetic field completely, (2) lift the mass against gravity, and (3) drive material motion at the observed speed.

### 3.4 Aly-Sturrock Paradox

*Barnes and Sturrock* [1972] numerically studied a process during which a closed force-free arcade system underwent a sudden, dynamic transition to a completely open field. This kind of transition, or something close to it, is needed for models of CMEs. From their analysis they concluded that a complete closed field could contain more magnetic energy than a completely opened field, and, therefore that the transition from closed to open was energetically favorable. However, *Aly* [1984] re-investigated their model and noticed that they had used cylindrically symmetric boundary condition that confined the magnetic field in a cylinder whose boundary behaved as a rigid conducting surface on which an induced current was produced as the system evolved. After removing this unreasonable boundary
condition, Aly found that the open state now contained more magnetic energy than the closed one, so the transition was no longer possible. Later, Aly [1991], and Sturrock [1991] established that for a simply connected field, the fully opened field configuration always has a greater magnetic energy than the corresponding force-free field. Aly [1991] also showed that for a simply connected force-free magnetic field the ratio of the total magnetic energy to the potential magnetic energy is necessarily less than 2. For example, the maximum ratio for the configuration with a Sun-centered dipole is about 1.66 [Low and Smith 1993; and Mikić and Linker 1994]. So, opening field lines means increasing the magnetic energy in the system, but storage models of CMEs require it to decrease [Sturrock et al. 1984]. This apparent contradiction is sometimes called the Aly-Sturrock paradox or Aly-Sturrock constraint.

The Aly-Sturrock constraint has confounded many proponents of storage models because it seems to imply that such models are energetically impossible. However, there are several ways to avoid this constraint. First, the magnetic field may not be simply connected and may contain knotted field lines; second, it may contain field lines that are completely disconnected from the photosphere; third, an ideal MHD eruption can still extend field lines provided it does not open them all the way to infinity; fourth, an ideal MHD eruption may be possible if it only opens a portion of the closed field lines; fifth, deviations from a perfectly force-free initial state might make a difference; and finally, a non-ideal MHD process, especially magnetic reconnection, might be important.

When studying a specific model, all of these possibilities, or some of them, are generally considered simultaneously. According to Forbes [2000], one can distinguish four different classes of eruption models. First is a class of non-force-free models which supposes that the gravity and gas pressure may play an important role in the storage of enough energy
and the initiation of an eruption. Second is a class of force-free models that attempts to explain the eruption solely in terms of an ideal MHD process. Third is a class of models that invokes resistive MHD processes such as magnetic reconnection to trigger the eruption. Finally, is a fourth class of hybrid models that initiate the eruption by a purely ideal MHD process but require the non-ideal MHD process of magnetic reconnection in order to sustain the eruption.

### 3.5 Non Force-Free Models

For the first class, a two-step-process is usually considered [Low 1990 and 1997; and Hundhausen 1999]. First, an initially closed coronal magnetic field is opened up and the mass previously trapped in the closed field is ejected. Second, the opened field lines are re-closed by the magnetic reconnection and result in a flare (see Hirayama [1974]; and Kopp and Pneuman [1976]). The first process is an ideal MHD one, and the second process is non-ideal MHD and produces the intense heating characteristic of the flare [Hiei et al. 1994 and 1997; and Low 1994].

In this scenario, gravity is used to bypass the Aly-Sturrock constraint. If a force-free field is confined in a fixed volume of space by rigid walls, its energy can grow without bound when subject to ever increasing stress that compresses the field against the walls. Low and Smith [1993] suggested that although there are no rigid walls in the solar atmosphere, the weight of plasma in a non-force-free magnetic field acts like a rigid wall to confine the magnetic field. Low [1999] suggested that the weight of quiescent prominence serves to hold the magnetic field (which he supposes is in the form of a flux rope) in place, much like a weight on top of a spring [Klimchuk 2001]. Forbes [2000] estimated that gravitational energy could allow the stored magnetic energy to exceed its maximum force-free value by
as much as 10%.

Some of the cool plasma in an erupting prominence is often seen to fall back to the surface, which suggests that a CME might be triggered if the magnetic field slowly evolves to a critical point where it can no longer support the prominence [Low 1996, 1997 and 1999]. In other words, the weight of the prominence would act as a lid that allows the magnetic energy to increase above the open limit, and when the lid is suddenly removed, the field springs outward. However, many CMEs do not appear to contain any prominence material, so it seems unlikely that such a mechanism could explain all CMEs. If both gas pressure and gravity are included [Low and Smith 1993; Wolfson and Dlamini 1997; and Wolfson and Saran 1998], gas pressure reduces the magnetic energy that can be stored in the corona [Low 1999 and Forbes 2000]. But unlike gravity, gas pressure can itself propel material outward given the appropriate gradient. So far, all models which invoke gas pressure, even if only as a trigger, continually run into the problem that the plasma $\beta$ (the ratio of gas pressure to magnetic compression) of the low corona is too small ($10^{-4}$ to $10^{-3}$) for gas pressure to have a significant effect.

3.6 Ideal MHD Models

The second class of models are based on the purely ideal MHD. This class deals with processes during which no dissipation or diffusion of the magnetic field occurs. Although the magnetic reconnection can occur, it is assumed to play no fundamental role in the triggering or long term evolution of the system. These models are severely restricted by the Aly-Sturrock constraint. But one way to elude the constraint is to assume that only a portion of the total field is opened by eruption [Wolfson 1993]. Wolfson and Low [1992] found a partly open field that has a lower magnetic energy than a fully closed field with
the same photospheric boundary condition. However, their method of solution does not allow them to determine whether a closed state may transit into an open state without invoking reconnection. So, it is still not clear that whether a partly open magnetic field can be achieved solely by a loss of ideal MHD equilibrium [Forbes 2000].

3.7 Ideal-Resistive Hybrids

3.7.1 Sheared Arcade Model

In simply-connected force-free magnetic arcades, the free energy continually increases as the footpoints of the arcade are sheared. Here, "simply-connected" means that the two ends of all field lines are anchored in the photosphere. As the field is sheared a stable equilibrium exists, so that a sudden dynamic process never occurs, and therefore, simply shearing an arcade cannot reproduce the eruptive behaviors of flares and CMEs [Finn and Chen 1990]. Several existing numerical simulations [Linker and Mikić 1994; and Mikić and Linker 1994] show that the simply-connected magnetic arcades on a spherical surface do not erupt, but instead expand outward smoothly with increasing shear of the footpoints until a fully open field is formed when the shear exceeds a critical value. As the field expands, a current sheet develops that separates regions of opposite magnetic polarities, but no eruption occurs and the whole system always remains in equilibrium.

One way to force an eruption is to suppose that the current sheet suddenly allows reconnection to take place at some point in time. For example, we might assume that once the current sheet is thin enough, it is subject to the tearing mode instability [Furth et al. 1963]. Alternatively, a microinstability (i.e. a phase spare instability) may occur when the current density in the sheet exceeds some threshold value [Galeev and Zelenyi 1975;
Figure 3-1: (a) Quasi-static evolution of an axially symmetric arcade which is sheared by rotating the northern and southern hemispheres of the Sun in opposite directions. The initial field at $t = 0$ is a Sun centered dipole which evolves into the force-free field shown at $t = 540 \tau_A$. After a rotation of $126^\circ$, the field becomes fully opened at $t = 900 \tau_A$, as long as the magnetic resistivity, $\eta$, remains zero. However, an eruption occurs at $t = 563 \tau_A$ if $\eta$ is suddenly increased. (b) The corresponding evolution of total energy divided by the potential energy (from Forbes [2000]).
and Heyvaerts and Priest 1976]. Once a microinstability occurs, the resistivity suddenly increases, which causes the magnetic field lines to reconnect in the current sheet, which, in turn, leads to the formation of an island as shown in Figure 3-1.

Prior to the eruption, the reconnection rate must be much slower than the rate at which the photospheric motions stresses the field, but once eruption occurs, the reconnection rate must be fast to dissipate the energy rapidly enough [Lin and Forbes 2000; and Forbes 2000].

3.7.2 Break-Out Model

Antiochos et al. [1999] proposed another sheared arcade model that also requires magnetic reconnection to trigger the eruption. In this model the magnetic field configuration has a spherically symmetric quadrupolar geometry, rather than a dipolar geometry. In this model, there are three polarity inversion lines on the photosphere and four distinct flux systems as shown in Figure 3-2. There is a central arcade straddling the equator, two arcades associated with the neutral lines at ± 45° latitude, and a dipolar flux system overlying the three arcades. Unlike what had been done by Mikić and Linker [1994], Antiochos et al. [1999] chose to shear only the central arcade. This results in a rising central arcade which compresses the X-line above it to produce a curved, horizontal current layer on the top of the sheared arcades (Figure 3-2). In the absence of gas pressure or resistivity this layer is an infinitely thin sheet, and it confines the central arcade so that the latter cannot open without reconnection occurring in this sheet. They claimed that when gas pressure and reconnection are included, the reconnection at the X-line undergoes a sudden transition from slow to fast. Because of the gas pressure the current sheet has a finite thickness, but as the underlying sheared arcade presses against the sheet, it thins. In the numerical simulations this thinning appears to accelerate the reconnection considerably, although it
is difficult to say this conclusively since the sheet is no longer numerically resolved at the moment the acceleration occurs.

The difference of the break-out model from the sheared arcade model of Mikić and Linker [1994] relies on the fact that the break-out model requires that magnetic reconnection occur on the top of the sheared arcade. Although the trigger mechanism in the breakout model remains problematic, the model does clearly demonstrate rigorously a transition from a closed magnetic field to a partly open one that is energetically favorable.

### 3.7.3 Early Flux Rope Models

In 1979, W. Van Tend and M. Kuperus [Van Tend and Kuperus 1978; and Van Tend 1979] proposed a simple catastrophe model for CMEs suggesting that a coronal flux rope
(represented in their model by a simple line current) could lose equilibrium when its current exceeds a critical value (Figure 3-3). In their model the flux rope floats in the corona under a balance between magnetic compression, produced by the magnetic field below the flux rope, and the magnetic tension, produced by the magnetic field overlying the flux-rope (Figure 3-3a). In most circumstances, this balance is stable, if the flux-rope is perturbed, the rope just simply oscillated up and down around its equilibrium location (Figure 3-3b). As the flux-rope current increases, Van Tend [1979] showed that this stable equilibrium would be displaced upwards (Figures 3-3c and 3-3d), and that this displacement could occur either continuously or discontinuously depending on the nature of the background field. When the background field in their model falls off with height faster than $1/y$, where $y$ is vertical coordinate, the transition becomes discontinuous as the current exceeds a critical value. In order for the field to decrease with height in this fashion, the background field must be dominated by a two-dimensional dipole, or higher order multi-polar component. If the background field is dominated by a two-dimensional monopole (i.e. a line-current), then there is no sudden transition.

Subsequent studies by Kaastra [1985], Molodenskii and Filippov [1987], and Martens and Kuin [1989] generalized the model of Van Tend and Kuperus within the framework of circuit theory. (Despite the fact that they refer to their model as a “circuit model”, Martens and Kuin’s model is more like an MHD model except for their treatment of the dissipation processes associated with reconnection and shock waves.) In circuit theory, the flux rope (or current filament as they used) is simply treated as a wire immersed in a vacuum, and the magnetic field is not frozen to the plasma as it should be in the nearly ideal MHD environment of the corona. Consequently, reconnection occurs freely at the neutral point (or X-line) in their model. However, in a realistic coronal plasma environment, reconnection
Figure 3-3: Schematic diagrams showing the behavior of the magnetic energy in models of the Van Tend and Kuperus type. The shaded circle designates the flux rope, and solid arrows indicate flux rope motion. Hollow arrows show the photospheric convection which increases the current inside flux rope by reconnecting field lines in the photosphere (from Forbes and Isenberg [1991]).
Figure 3-4: Magnetic configuration of an analytical MHD model with a two-dimensional dipole located on the surface (from Forbes [1990a]). The flux rope and the upper and lower tips of the current sheet are located at $h$, $q$, and $p$, respectively.

is not easy because of the high conductivity of the plasma. Thus, any attempt to quickly change a configuration with an X-point in it leads to the formation of a current sheet at the X-point. Kaastra [1985] and Martens and Kuin [1989] addressed this problem by incorporating current sheets within the circuit framework, but the approximations they used restricted their analysis to unrealistically short current sheets (cf. Lin and Forbes [2000]) and the current in the sheet is always much weaker than the current in the flux rope.

Following upon the work of Kaastra [1985] and Martens and Kuin [1989], Priest and Forbes [1990] found an exact solution for a configuration with a current sheet of arbitrary length (Figure 3-4). However, this configuration has a background field (the potential field produced by the photospheric sources) with a two-dimensional dipole located on the surface, and the location of the source on the boundary means there is an unphysical singularity in
the field at this location. Nevertheless, it is possible with this model to analyze the effects of a finite current sheet of arbitrary length. The magnetic field in this model is given by [Priest and Forbes 1990 and Forbes 1990a]

\[
B_y + iB_x = i \frac{mh^2 \sqrt{(z^2 + p^2)(z^2 + q^2)}}{pqz^2(z^2 + h^2)},
\]

(3.1)

where \( i = \sqrt{-1} \), \( z = x + iy \), \( h \) is the height of the flux rope, \( p \) and \( q \) are the lower and upper end points of the current sheet, and \( m \) is the dipole strength. Applying the frozen-flux condition related the relative strength of dipole \( m/I \) to \( h, p, \) and \( q \) such that

\[
m/I = 2hpq \sqrt{(h^2 - p^2)(h^2 - q^2)},
\]

(3.2)

where \( I \) is the current inside the flux rope. In the absence of a current sheet and when the background dipole is placed at a depth, \( h_b \), below the photosphere, equilibria occurred at

\[
\frac{m}{I} = \frac{(h + h_b)^2}{2h}
\]

(3.3)

as shown in Figure 3-5 for \( h_b = 1 \). There is no equilibrium when \( m/I \) is less than the critical value, \( 2h_b \), but when the dipole is on the surface \( (h_b = 0) \), the critical value is zero. Furthermore, the lower branch of the equilibrium curve disappears into the singularity of the \( x \)-axis, so the bifurcation of the system is no longer physically meaningful.

As in Van Tend and Kuperus' model, the flux rope current in the works of Priest and Forbes [1990] and Forbes [1990a] was still used as an independent variable. As Isenberg et al. [1993] pointed out, the current inside the flux rope does not represent any fundamental factor that governs the system. Its value cannot be set independently, but must be determined
Figure 3-5: Equilibrium heights of the flux rope as a function of the ratio of the relative dipole strength. The dashed curves denote unstable equilibria, and $h_b$ is the dipole depth below the surface in arbitrary units (from Forbes [1990a]).

from the boundary conditions. Thus, simply showing that there is a nose point in the equilibrium curve when the flux rope height is plotted as a function of its current is not sufficient to prove the existence of a catastrophe. This has been a cause for much confusion in the past, even though, as we will see in the next chapter, the real critical point is not very far from that turning point in the height versus current curve.

### 3.8 Ideal-MHD Flux Rope Models

According to the theory developed by Thom [1972], any process, like a flare or CME, which involves a rapid transition from a steady, or quasi-steady, state to a dynamic state constitutes a catastrophe. Based on the previous ideas of Van Tend and Kuperus [1979], Kaastra [1985], Martens and Kuin [1989], and Van Ballegooijen and Martens [1989], Forbes and Isenberg [1991] developed a two-dimensional ideal-MHD force-free flux rope model for eruptions. In this model, a current-carrying flux rope is nested within an arcade, and the
reconnection in the corona is prohibited, so that a current sheet can form prior to the onset of the eruption. The frozen-flux condition at the flux-rope surface is used to determine the current flowing within the rope as a function of the boundary conditions at the photosphere. In the absence of significant pressure gradients or gravity, the equilibrium configuration in the corona is governed by the static MHD equations for a force-free field:

\[ j \times B = 0, \tag{3.4} \]
\[ j = \frac{c}{4\pi} \nabla \times B, \tag{3.5} \]

where \( B \) and \( j \) are the magnetic field and the current density, respectively. In a two-dimensional system with translational symmetry, these two equations can be combined and further simplified into the Grad-Shafranov equation

\[ \nabla^2 A + \frac{1}{2} \frac{dB_z^2}{dA} = 0 \tag{3.6} \]

in the semi-infinite \( x-y \) plane with \( y \geq 0 \), where \( B_z \) is the field perpendicular to the \( x-y \) plane and \( A(x, y) \) is the flux function defined by

\[ (B_x, B_y, B_z) = \left[ \frac{\partial A}{\partial y}, -\frac{\partial A}{\partial x}, B_z(A) \right]. \tag{3.7} \]

The surface at \( y = 0 \) corresponds to the photosphere. Equation (3.6) is then usually used to construct an evolutionary sequence of force-free equilibria in response to quasi-static changes in the photosphere, i.e., changes that occur on time-scales much greater than the time for magnetoacoustic waves to traverse the configuration.

The equilibrium magnetic configuration is obtained by setting the total force acting
on the flux rope to zero. In a two-dimensional force-free configuration with translational symmetry, the total force per unit length acting on the flux rope is

$$F_{tot} = \frac{1}{c} \int \int_S (j_f \times B_f) da + \frac{1}{c} \int \int_S (j_f \times B_e) da,$$

(3.8)

where $S$ indicates the region occupied by the rope and $a$ is the area of $S$, $j_f$ is the current density inside the flux rope, and $B_f$ and $B_e$ are the magnetic fields due to the internal current of the flux rope and the external current outside the flux rope, respectively. Generally, the area $a$ and radius $r_0$ of the flux rope are assumed to be small enough to make the external field $B_e$ effectively uniform within the flux rope. This assumption dissociates the condition for equilibria to into two separate conditions: one for the internal, local equilibrium, given by

$$j_f \times B_f = 0,$$

(3.9)

and one for the external, global equilibrium, given by

$$F = \frac{I B_e}{c} \hat{y} = 0,$$

(3.10)

where $F$ is the external force per unit length, $B_e$ is the external field evaluated at the center of the flux rope, and $I$ is the current strength inside the flux rope,

$$I = \int \int_S j_f da.$$

(3.11)

Since $I$ is not zero, equation (3.10) requires

$$B_e = 0.$$

(3.12)
This condition prescribes the global equilibria in response to the slow change of the boundary conditions at the photosphere. The adjustment of the flux rope radius, \( r_0 \), and area, \( a \), is prescribed by the internal equilibrium condition (3.9) which establishes the relation of \( r_0 \) to \( I \).

The evolution of this system generally takes two stages: the first stage stores energy, while the second releases it. During the storage phase, the convective motion of photospheric material builds up stress in the coronal field and leads to the storage of magnetic energy. The evolution is so slow (with a time-scale of photospheric motions) that it can be regarded as quasi-static. During the second, eruptive phase, mechanical equilibrium is lost, the system evolves rapidly over an Alfvén time scale. It is during this second phase that the flux rope is ejected upward and that the magnetic energy stored in the system is released. Since the evolution during the ejection is much faster than the rate at which the energy is transferred from the photosphere to the corona, the transfer of energy from the photosphere to the corona is completely negligible during the eruption itself.

### 3.8.1 Boundary Conditions, Energetics and Critical Radius

For their model, Forbes and Isenberg [1991] chose the boundary condition

\[
A(x, 0) = \frac{md}{x^2 + d^2} - \phi(t),
\]

on the photospheric surface, where \( m \) and \( d \) are constants and \( \phi(t) \) is a slowly varying function of time. This boundary condition is the same as that produced by a line dipole of strength \( m \) located at a depth of \( d \) below the photosphere. The function \( \phi(t) \) governs the magnetic flux transported from the photosphere to the corona through reconnection at the point \((0, 0)\), and thus it parameterizes the storage of magnetic energy.
With the reconnection occurring at point (0, 0), magnetic flux, \( \phi(t) \), is successively transported into the coronal field and the system evolves along the equilibrium curve given in the lower panel of Figure 3-6. This figure shows four different stages in an evolutionary sequence. Panels 1 through 3 in the upper part of Figure 3-6 show the quasi-static evolution along the equilibrium curve from a nearly potential configuration up to the catastrophe-point configuration. Because reconnection is forbidden in this model, a short current sheet attached to the base develops after an X-point appears prior to the loss of equilibrium.

In the zero-\( \beta \) limit (corresponding to a strong magnetic field), the current sheet in Figure 3-6 is infinitesimally thin, so the flux reconnected at the photosphere immediately appears at the top of the current sheet. For a sufficiently small flux rope radius (less than one-thousandth of the scale length), the equilibrium curve becomes multi-valued with a turning, or nose point as shown in the bottom panel. As mentioned previously, a multi-valued equilibrium curve with a turning point does not necessarily prove the presence of a catastrophe since one also must show that the system can actually be driven beyond this point by the evolution of the boundary conditions. Because Forbes and Isenberg [1991] used \( \phi \), which is an independent variable and embodies the changes occurring at the photosphere, the equilibrium curve truly describes the evolution of the system in response to the change of the boundary conditions. Thus, the turning point is a physically meaningful one at which the system undergoes an abrupt transition from the lower equilibrium (panel 3) to the upper equilibrium (panel 4).

However, this model treated the flux rope as an incompressible line current with a constant finite radius, while the plasma exterior to the flux rope was highly compressible (corresponding to plasma \( \beta \to 0 \)). So, this model did not treat the regions interior and exterior to the flux rope in a self-consistent manner.
Figure 3-6: Magnetic configurations at various stages in the analytical solution of Forbes and Isenberg [1991]. A catastrophic loss of equilibrium occurs when the evolution reaches the critical configuration of panel 3. The bottom panel shows the equilibrium filament height as a function of the stored magnetic energy. The dashed line is the expected flux rope trajectory when the system reaches the critical point (from Forbes [1991]).
3.8.2 System with Quadrupolar Boundary Condition

The limitation of the *Forbes and Isenberg*’s [1991] model encouraged them to investigate more realistic situations. Of most concern, was the fact that the critical radius, $r_c$, is unrealistically small, being less than $10^{-3}$ times the global scale length of the configuration. This small radius implies an unrealistically strong magnetic field on the surface of the flux rope in excess of $10^5$ Gauss [*Forbes and Isenberg* 1991], whereas the actual field is very unlikely to exceed $10^3$ Gauss.

To improve their model, *Isenberg et al.* [1993] replaced the incompressible flux rope with a proper force-free magnetic flux rope whose current and radius are self-consistently determined from the quasi-static MHD equations following the prescription given by *Parker* [1974]. The dipolar boundary condition was also replaced with a quadrupolar boundary condition in the expectation that this would give a more reasonable radius for catastrophic behavior. In the quadrupole model, the boundary condition (3.13) becomes

$$A(x, 0) = \frac{m(d^2 - x^2)}{(d^2 + x^2)^2}, \quad (3.14)$$

where $d$ is a constant and $m$ is a slowly varying function of time. This boundary condition represents a field equivalent to that produced by a two-dimensional magnetic quadrupole of strength $m$ below the photosphere at a depth $d$. As we can see, the boundary condition (3.14) corresponds to a magnetic field which falls off with height as $y^{-3}$, faster than the two-dimensional dipolar field which falls off with height as $y^{-2}$. This difference improves the catastrophe behavior of the system. The critical radius required for catastrophe increases. The previous model exhibited a catastrophic loss of equilibrium only for radii less than the implausibly small value of $10^{-3}$ times the global scale length. In the quadrupole model, on
the other hand, a loss of equilibrium can occur for radii less than 0.23 times of the global scale length — more than two orders of magnitudes larger than that in the previous model. The magnetic field at the surface of the flux rope is now less than 500 Gauss, still somewhat high.

Another important output from the second model is that it established that ideal-MHD catastrophes do exist in force-free magnetic configurations. Before Forbes and Isenberg [1991], Klimchuk and Sturrock [1989] had questioned whether a purely force-free field can ever exhibit an ideal MHD catastrophe. Upon re-examining an earlier force-free configuration by Low [1977], they found that the catastrophe-like behavior which occurred in that configuration was an artifact of the solution method used by Low [1977]. So, Sturrock [1989] was led to speculate that only non-force-free system can exhibit catastrophic behavior, but Isenberg et al. [1993] confirmed that a purely force-free field still can exhibit catastrophic behavior.

3.8.3 System with Two-Point Source Boundary Condition

Encouraged by the results from quadrupole model, Forbes et al. [1994] used variational calculus methods to determine the optimal boundary condition for catastrophe behavior, i.e., a boundary condition which would maximize the released magnetic energy by an ideal-MHD catastrophe. They found that for a simply-connected background field (namely all field lines cross the axis of symmetry and root in the photosphere as in a simple arcade), the field produced by two point magnetic charges gave the greatest energy release as a percentage of the total free energy of the system. In terms of the vector potential, this
boundary condition is simply:

\[ A(x, 0) = A_0 H(\lambda - |x|), \]

(3.15)

where \( H \) is the Heaviside step function, \( A_0 \) is the net flux through the photosphere in the region \( x \geq 0 \) (or equivalently, the value of \( A \) at the origin), and \( \lambda \) is the distance between two point sources of opposite polarity. The equilibrium curve and the magnetic configurations at different evolutionary stages are shown in Figure 3-7.

Magnetic reconnection in all of the above models is prohibited, so, the magnetic tension associated with the current sheet which forms attached to the boundary surface can always be strong enough to prevent the flux rope from escaping. Also without reconnection, only a small fraction of the total stored magnetic energy is released during the ideal-MHD jump between the turning point and upper equilibrium state. The maxima of the fraction of energy released for the boundary conditions of dipole, quadrupole, hexapole, and two point source are 0.99%, 5.78%, 8.33%, and 8.58%, respectively. Forbes [1994] also considered the maximum energy which can be released for non-simply connected background field, and found that the energy which could be released could, in principle, be as large as 20.76% of the total free energy.
Figure 3-7: (a) Flux-rope height, $h$, as a function of the separation half-distance, $\lambda$, between the photospheric sources. The dashed section of the curve indicates the region which is bypassed because of the jump at the critical point. Figures (b), (c), and (d) show magnetic configurations in the $x$-$y$ plane at the three locations indicated in (a) (from Forbes and Priest [1995]).
Chapter 4

The Effects of Reconnection on the Dipole Model

In this chapter, we present the first of new research projects that improves the models by including physically important effects that were previously ignored. The first of these effects is magnetic reconnection.

4.1 Introduction

If reconnection is prohibited, it is impossible for the flux rope to escape to infinity in the two-dimensional models. Whether this is still true for three dimensional models remains unknown. Escape is prevented by the magnetic tension associated with the development of a current sheet attached to the boundary surface. For some specific boundary conditions, this attached current sheet may form before the loss of equilibrium (e.g. Forbes and Isenberg [1991] while for others it may form afterwards). Numerical experiments also show the development of this current sheet following a jump [Forbes 1990b and 1991]. However, due to the finite size of the numerical box, the simulation cannot run long enough to investigate the full evolution of the sheet. Thus, a simplified analytical solution should provide the information that a simulation cannot.

The electrical conductivity of the coronal plasma, although high, is not infinite. Therefore, dissipation of the magnetic field in the corona is not completely prevented. Generally,
the following inequality holds
\[ \tau_p \gg \tau_R \gg \tau_A, \] (4.1)

where \( \tau_p, \tau_R, \) and \( \tau_A \) are the time scales of photospheric motion, magnetic reconnection, and Alfvén wave propagation, respectively. Thus, during energy storage before the loss of equilibrium, no extensive current sheet can develop because the rate at which it is dissipated by reconnection is faster than the rate at which it is built up by photospheric motions. Once the equilibrium is lost, and the flux rope jumps upward, the evolution occurs at Alfvén time scale, \( \tau_A \), which is less than \( \tau_R \), so a current sheet must form once an X-line appears.

4.2 Construction of the Model

In order to include the reconnection process, it is necessary to consider a configuration with a current sheet that is detached from the surface. The analysis with a detached current sheet is much more complicated mathematically because it contains mixed Dirichlet and Neumann boundary conditions. One way to solve it is to transform the mixed boundary value problem into a singular integral equation that can be solved by using the method developed by Muskhelishvili [1953]. In general, a closed solution for this integral equation does not exist, but can be found in some specific cases.

4.2.1 Basic Description

The model we consider in this chapter consists of a two-dimensional magnetic configuration in the semi-infinite \( z-y \) plane with \( y = 0 \) being the photospheric boundary and \( y > 0 \) corresponding to the corona. At any given time \( t \) a force-free flux-rope with radius \( r_0 \) is located at height \( h \) on the \( y \)-axis. Below it there may exist a detached vertical current sheet.
along the $y$ axis with its lower tip at $y = p$ and its upper tip at $y = q$ as shown in Figure 3-4. As discussed previously, reconnection cannot be prohibited in the corona due to the finite plasma conductivity.

To deal with the dynamics of the eruption, we make the following assumptions:

1. The flux rope is treated as a projectile and the generation of MHD waves (e.g., a shock wave) by its motion is ignored.

2. All the magnetic energy released goes into accelerating the flux rope. Energy lost to radiation, heating, and chromospheric evaporation is ignored.

3. Gas pressure is negligible compared to magnetic pressure except in the reconnection region.

4. The only significant currents in the corona are in the flux rope and the reconnecting current sheet of zero thickness.

5. Gravitation is ignored. (This is done for simplicity.)

6. The reconnection rate, as prescribed by the inflow Alfvén Mach number $M_A$ at the midpoint of the current sheet, is given. In the following work, we will assume that $M_A$ is constant and in the range between 0 and 1, but more complicated models, such as those due to Sweet-Parker [Sweet 1958; Parker 1957 and 1963] and Petschek [1964], can easily be incorporated.

The errors generated by the simplifying assumptions (1) through (4) can be roughly estimated by comparing the results of our model with the numerical simulation by Forbes [1991], who used the full set of compressible, resistive MHD equations to test the flux rope model. The simulation shows that only about half of the released magnetic energy is actually converted into the kinetic energy of the flux rope, and the other half goes into heating, radiation, and the generation of MHD waves. Thus our model overestimates the
speed at which the flux rope moves, but for any given rate of reconnection, $M_A$, the model does provide an upper limit on how fast the stored magnetic energy can be released.

### 4.2.2 Formulation

In our formulation, we do not address the question of where the prominence material is located relative to the current carrying region that constitutes the flux rope. Although it may be that the two coincide, it is more likely that they do not. Observations show that the upward motion of the prominence usually lags the upward motion of the overlying region by several minutes. This lag suggests that the $j \times B$ force that accelerates material upward first appears at altitudes higher than the prominence itself [Hundhausen 1988].

If our model problem were quasi-static, rather than dynamic, assumption (4) would automatically follow from assumption (3). However, since this is not the case, the use of a quasi-static type solution to model the current sheet constitutes an additional assumption. Because we assume the rate of reconnection to be given (assumption 6), a detailed description of the flow and field within the current sheet is not needed. All that is required is an estimation, at the location of the flux rope, of the component of the field due to the current in the sheet.

With the above assumptions, the magnetic configuration in our system is still described by either equations (3.4) and (3.5), or equations (3.6) and (3.7), with the appropriate boundary conditions. Usually, equation (3.6) is re-written as

$$\nabla^2 A = -\frac{4\pi}{c} j_z(x, y), \quad (4.2)$$

where $j_z$ is the $z$ component of the current density in the corona. This current density consists of two parts: that inside the flux rope, $j_f(z, y)$, and that inside the current sheet,
\( j_c(x, y) \), such that

\[
j_z(x, y) = j_f(x, y) + j_c(x, y).
\] (4.3)

Since the radius of the flux rope is small, the magnetic field is approximately the same as that produced by a line current of strength \( I \) located at \((0, h)\), and the external field on the flux rope can be approximately evaluated at the center of the flux rope, \((0, h)\). Therefore, we have

\[
j_f(x, y) = I(h)\delta(x)\delta(y - h),
\] (4.4)

provided \(|y - h| \gg r_0\). Here \( \delta \) is the Dirac delta-function. As for the current sheet, its thickness is small compared to the scales in other directions, so we have

\[
j_c(x, y) = K_S(y)\delta(x)[H(y - p) - H(y - q)],
\] (4.5)

where \( H \) is the Heaviside step-function, and \( K_S \) is the surface current density in the current sheet which extends from the lower tip \( y = p \) to the upper tip \( y = q \).

### 4.2.3 Formal Solution for the Flux Function

A general form of solution for equation (4.2) in domain \(-\infty < x < \infty, 0 < y < \infty\) is

\[
A(x, y) = \frac{1}{c} \int_{-\infty}^{\infty} \int_0^\infty G(x, y; u, v)j_z(u, v)dv du + \frac{1}{4\pi} \int_{-\infty}^{\infty} A(u, 0) \frac{\partial G}{\partial v} |_{v=0} du,
\] (4.6)

where \( G(x, y; u, v) \) is the two-dimensional Green's function for the Dirichlet problem:

\[
G(x, y; u, v) = \ln \left[ \frac{(x - u)^2 + (y + v)^2}{(x - u)^2 + (y - v)^2} \right],
\] (4.7)
which satisfies

\[ G(x, 0; u, v) = G(x, y; u, 0) = 0. \]

Substituting equations (4.3), (4.4), (4.5), and (4.7) into equation (4.6) gives

\[
A(x, y) = \frac{I_0}{c} \left\{ \int \ln \left[ \frac{x^2 + (y + h)^2}{x^2 + (y - h)^2} \right] + \int_p^q \ln \left[ \frac{x^2 + (y + v)^2}{x^2 + (y - v)^2} \right] \kappa_S(v) dv \right\} + \frac{1}{\pi} \int_{-\infty}^\infty \frac{A(u, 0)y du}{(x-u)^2 + y^2}, \tag{4.8}
\]

where \( I_0 \) is a constant and has the dimension of current intensity, \( J = I/I_0, \kappa_S(y) = K_S(y)/I_0, \) and \( A(x, 0) \) is the boundary condition on the photosphere.

We require that the transverse field along the current sheet vanish, namely \( B_x(0, y) = \partial A(0, y)/\partial y = 0 \) for \( p \leq y \leq q \). Taking the partial derivative on \( A(x, y) \) in (4.8) with respect to \( y \) and setting \( x = 0 \), we get

\[
B_x(0, y) = \frac{\partial A(0, y)}{\partial y} = \frac{I_0}{cd} \left\{ \frac{4Jh}{h^2 - y^2} + \int_p^q \frac{4v \kappa_S(v) dv}{v^2 - y^2} \right\} + \frac{1}{\pi d} \int_{-\infty}^\infty \frac{u^2 - y^2}{(u^2 + y^2)^2} A(u, 0) du. \tag{4.9}
\]

Setting \( B_x(0, y) = \partial A(0, y)/\partial y = 0 \) for \( p \leq y \leq q \) gives

\[
\int_p^q \frac{4v \kappa_S(v) dv}{v^2 - y^2} = \frac{c}{I_0 \pi} \int_{-\infty}^\infty \frac{y^2 - u^2}{(u^2 + y^2)^2} A(u, 0) du - \frac{4Jh}{h^2 - y^2} \quad p \leq y \leq q.
\]

Integrating by parts over the first term at the right hand side of this equation leads to

\[
\int_p^q \frac{4v \kappa_S(v) dv}{v^2 - y^2} = -\frac{c}{I_0 \pi} \int_{-\infty}^\infty \frac{u}{u^2 + y^2} \frac{\partial A(u, 0)}{\partial u} du - \frac{4Jh}{h^2 - y^2} \quad p \leq y \leq q, \tag{4.10}
\]
which is a singular integral equation about $\tilde{K}_S(y)$. *Muskelishvili* [1953] analyzed this class of equations in detail, and *Xu* [1992] outlined the general method for applying *Muskelishvili's* analysis. Solving equation (4.10) is usually very difficult and tedious. Whether a closed form for $\tilde{K}_S(y)$, as well as one for $A(x, y)$ in (4.8) exists depends on the functional behavior of $A(x, 0)$.

### 4.3 Effects of Reconnection on the Critical Radius

*Xu* [1992] did not actually find a solution for (4.8) in a closed form due to the fact that he chose boundary conditions that made it difficult to complete the essential integrals. In our work, we are extending *Xu's* [1992] work to as close to the final form as possible. First of all, considering the inequality (4.1), we are re-calculating the critical radius of the flux rope for the dipolar boundary condition.

As *Xu* [1992] did, we still use the boundary condition (3.13) which corresponds to a two-dimensional dipole [*Forbes and Isenberg* 1991]. For simplicity, we use the depth $d$ as the length scale and all lengths in the calculation are normalized to it. We also normalize the dipole strength $m$ to the current strength $I_0$ such that $mc = 4I_0d$, where $c$ is the light speed. So, the boundary condition (3.13) becomes

$$A(x, 0) = \frac{I_0}{c} \frac{4}{x^2 + 1} - \phi(t). \quad (4.11)$$

Substituting (4.11) into (4.9) and (4.10) yields

$$B_x(0, y) = \frac{4I_0}{cd} \left[ \frac{J}{2} \left( \frac{1}{h-y} + \frac{1}{h+y} \right) + \int_p vK_S(v)dv + \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{u^2 - y^2}{(u^2 + y^2)^2} \frac{du}{u^2 + 1} \right], \quad (4.12)$$

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respectively. The first term in the parentheses of equation (4.12) is the contribution from the current inside the flux rope. Subtracting this term from (4.12) gives the external field acting on the flux rope mentioned in (3.10):

\[
B_e = \frac{4I_0}{cd} \left[ \frac{J}{4h} + \int_p^q \frac{v\tilde{K}_S(v)dv}{y^2 - v^2} + \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{u^2 - y^2}{(u^2 + y^2)^2} \frac{du}{u^2 + 1} \right].
\]

(4.14)

Setting \(B_e = 0\) determines the global equilibrium.

Before the loss of equilibrium, no current sheet exists due to (4.1). Setting \(B_e = 0\) and \(\tilde{K}_S = 0\) in the relevant equations above gives

\[
\frac{J}{2h} - \frac{2}{(h + 1)^2} = 0.
\]

(4.15)

The corresponding flux between the flux rope surface and the photosphere is \([Forbes and Isenberg 1991]\):

\[
\phi = \frac{2I_0}{c} \left[ J \ln \left( \frac{2h}{r_0} \right) - \frac{2h}{h + 1} \right],
\]

(4.16)

where \(r_0\) is the radius of the flux rope. This radius was treated as a constant by \(Forbes and Isenberg [1991]\), but it is not consistent in a compressive plasma and must be adjusted according to the current inside the flux rope. The radius, \(r_0\), is related to \(J\) by

\[
r_0 = \frac{r_00}{J},
\]

(4.17)
where \( r_{00} \) is the value of \( r_0 \) at \( J = 1 \). This is actually a rough approximation of Parker's [1974] exact solution. The accuracy of this approximation was discussed in detail by Forbes and Priest [1995]. Combination of equations (4.15), (4.16) and (4.17) determines the equilibrium height, \( h \), of the flux rope as a function of \( \phi \).

To check if a catastrophe may occur in a system with reasonable parameters, we eliminate \( J \) and \( r_0 \) from (4.16) by using (4.15) and (4.17), and then set \( d\phi/dh|_{h=h_c} = 0 \), where \( h_c \) is the height of flux rope at the critical point. This leads to

\[
\ln \left[ \frac{8h_c^2}{(h_c + 1)^2r_{00}} \right] = \frac{3 - h_c}{2(h_c - 1)},
\]

(4.18)

and for very small \( r_{00} \) we have

\[
h_c = 1 + \frac{1}{\ln(2/r_{00})} - \frac{1}{2\ln^2(2/r_{00})} + \cdots.
\]

(4.19)

Consequently,

\[
J_c = 1 - \frac{1}{4\ln^2(2/r_{00})} + \cdots.
\]

(4.20)

So, \( h_c \) is always larger than unity, \( J_c \) is always less than unity, and both approach unity as \( r_{00} \) goes to zero. The behaviors of \( h_c \) and \( J_c \) are similar to those in the case studied by Forbes and Isenberg [1991]. Figure 4-1 shows the \( h \) versus \( \phi \) curve for the dipole field with an X-line, rather than a current sheet, prior to eruption. The reason why a plausible catastrophe can still occur in the case of large \( r_{00} \) is that no current sheet attached to the base develops during quasi-static phase of the evolution. Therefore, the tension force associated with the current sheet is not strong enough to prevent a catastrophe from happening for any value of \( r_{00} \). Thus, a critical radius no longer exists when we assume that \( \tau_R \ll \tau_p \), instead of
Figure 4-1: Equilibrium height, $h$, as a function of the reconnection flux, $\phi$, for $r_{00} = 3.7 \times 10^{-2}$. As the reconnected flux, $\phi$, increases, the filament moves along the lower branch of the equilibrium curve towards the turning point at point $\phi \approx 3$, which is also the critical point in this case. When the critical point is reached the flux rope is ejected upwards.

$\tau_R \gg \tau_p$.

4.4 The Dipole Configuration with a Detached Current Sheet

In this section, we consider the same field configuration as that of Forbes and Isenberg [1991] except that here the current sheet is detached from the base.

4.4.1 Solution for the Magnetic Structure

Xu [1992] already obtained the solution of equation (4.13):

$$K_S(v) = \frac{2}{\pi} \frac{Q(v)}{\sqrt{(v^2 - p^2)(q^2 - v^2)}} \quad p \leq v \leq q, \quad (4.21)$$

where

$$Q(v) = P_0 - \frac{Jh\sqrt{(h^2 - p^2)(h^2 - q^2)}}{h^2 - v^2} - \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{\sqrt{(u^2 + p^2)(u^2 + q^2)}}{(u^2 + v^2)(u^2 + 1)^2} u^2 du, \quad (4.22)$$
and $P_0$ is an integral constant to be determined. We can express $Q(v)$ in a closed form, but we leave the last term in (4.22) unintegrated in order to save space. Substituting (4.21) and (4.22) into (4.12) gives

$$B_x(0, y) = \frac{4I_0}{cd} \frac{1}{\sqrt{(p^2 - y^2)(q^2 - y^2)}} \left\{ P_0 - \frac{Jh\sqrt{(h^2 - p^2)(h^2 - q^2)}}{(h^2 - y^2)} \right.$$

$$- \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{\sqrt{(u^2 + p^2)(u^2 + q^2)}}{(u^2 + v^2)(u^2 + 1)^2} u^2 du \left\{ \begin{array}{ll} 1 & y < p \\ 0 & p \leq y \leq q \\ -1 & y > q \end{array} \right.$$

We noticed that $B_x(0, y)$ has singularities at the two tips of the current sheet, $(0, p)$ and $(0, q)$, unless

$$P_0 = \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{h^2 + u^2}{\sqrt{(u^2 + p^2)(u^2 + q^2)}} \frac{u^2 du}{(u^2 + 1)^2}$$

$$= \frac{2h^2}{\pi q(p^2 - 1)^2(q^2 - 1)} \left\{ \frac{p^2(1 - q^2)(1 + h^2 - 2p^2)K(k') - q^2(p^2 - 1)(h^2 - 1)E(k')}{p^2[p^2q^2 - 1] - p^2(3q^2 - 1) - 2q^2 + 1] \Pi(1 - p^2, k') \right\},$$

and

$$Jh = \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{(h^2 - p^2)(h^2 - q^2)}{\sqrt{(u^2 + p^2)(u^2 + q^2)}} \frac{u^2 du}{(u^2 + 1)^2}$$

$$= \frac{2\sqrt{(h^2 - p^2)(h^2 - q^2)}}{\pi hq(p^2 - 1)^2(q^2 - 1)} \left\{ p^2(p^2q^2 - 1)\Pi(1 - p^2, k') - p^2(q^2 - 1)K(k') - q^2(p^2 - 1)E(k') \right\},$$

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where \( k' = 1 - p^2/q^2 \), \( K, E \), and \( \Pi \) are the complete elliptic integrals of the first, second, and third kinds, respectively. These conditions force the field to have a configuration with a Y-type neutral point at each tip of the current sheet as shown in Figures 3-4, 4-2, etc.

Substituting (4.11), (4.21), (4.22), (4.24) and (4.25) into (4.8), we get, after some rather tedious algebra, the explicit expression for \( A(x, y) \):

\[
A(x, y) = \begin{cases} 
\frac{2I_0}{q} A_1(x, y) - \phi(t) & x^2 + y^2 \geq p^2 \\
\frac{2I_0}{q} A_2(x, y) + 4I_0/c - \phi(t) & x^2 + y^2 < p^2.
\end{cases}
\] (4.26)

where

\[
A_1 = \frac{P_0}{q} \left\{ F \left[ \sin^{-1} \left( \frac{q}{z_1}, \frac{p}{q} \right) \right] + F \left[ \sin^{-1} \left( \frac{q}{z_2}, \frac{p}{q} \right) \right] \right\} \\
- \frac{J}{hq} H_{PQ} \left\{ \Pi \left[ \sin^{-1} \left( \frac{q}{z_1}, \frac{p^2}{h^2 \cdot q} \right) \right] + \Pi \left[ \sin^{-1} \left( \frac{q}{z_2}, \frac{p^2}{h^2 \cdot q} \right) \right] \right\} \\
+ \frac{J}{2} \ln \left( \frac{hZ_1PQ + z_1H_{PQ}}{hZ_1PQ - z_1H_{PQ}} \right) \\
+ \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{udu}{(u^2 + 1)^2} \left( u(q^2 - p^2) \right) \left\{ \Pi \left[ \sin^{-1} \left( \frac{q}{z_1}, \frac{u^2 + p^2}{u^2 + q^2}, \frac{p}{q} \right) \right] \right\} \\
+ \Pi \left[ \sin^{-1} \left( \frac{q}{z_2}, \frac{u^2 + p^2}{u^2 + q^2}, \frac{p}{q} \right) \right] \\
- \frac{u}{q} \sqrt{u^2 + p^2} \left\{ F \left[ \sin^{-1} \left( \frac{q}{z_1}, \frac{p}{u^2 + p^2} \right) \right] + F \left[ \sin^{-1} \left( \frac{q}{z_2}, \frac{p}{u^2 + q^2} \right) \right] \right\} \\
+ \tan^{-1} \left\{ \frac{u}{z_1} \frac{(u^2 + p^2)(z_1^2 - q^2)}{(u^2 + q^2)(z_1^2 - p^2)} \right\} + \tan^{-1} \left\{ \frac{u}{z_2} \frac{(u^2 + p^2)(z_2^2 - q^2)}{(u^2 + q^2)(z_2^2 - p^2)} \right\} \right), \tag{4.27}
\]

\[
A_2 = \frac{P_0}{q} \left\{ F \left[ \sin^{-1} \left( \frac{z_1}{p}, \frac{p}{q} \right) \right] + F \left[ \sin^{-1} \left( \frac{z_2}{p}, \frac{p}{q} \right) \right] \right\} \\
- \frac{J}{hq} H_{PQ} \left\{ \Pi \left[ \sin^{-1} \left( \frac{z_1}{p}, \frac{p^2}{h^2 \cdot q} \right) \right] + \Pi \left[ \sin^{-1} \left( \frac{z_2}{p}, \frac{p^2}{h^2 \cdot q} \right) \right] \right\} \\
- \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{udu}{(u^2 + 1)^2} \left( u(q^2 - p^2) \right) \left\{ \Pi \left[ \sin^{-1} \left( \frac{z_1}{p}, \frac{p^2(u^2 + q^2)}{q^2(u^2 + p^2)}, \frac{p}{q} \right) \right] \right\}
\]
\[ + \Pi \left[ \sin^{-1} \left( \frac{z_2}{p} \right), \frac{p^2(u^2 + q^2)}{q^2(u^2 + p^2)}, \frac{p}{q} \right] \]

\[ + \frac{u}{q} \sqrt{\frac{u^2 + p^2}{u^2 + q^2}} \left[ F \left[ \sin^{-1} \left( \frac{z_1}{p} \right), \frac{p}{q} \right] + F \left[ \sin^{-1} \left( \frac{z_2}{p} \right), \frac{p}{q} \right] \right] \]

\[ + \tan^{-1} \left\{ \frac{z_1}{u} \sqrt{\frac{(u^2 + q^2)(z_2^2 - p^2)}{(u^2 + p^2)(z_1^2 - q^2)}} \right\} + \tan^{-1} \left\{ \frac{z_2}{u} \sqrt{\frac{(u^2 + q^2)(z_2^2 - p^2)}{(u^2 + p^2)(z_2^2 - q^2)}} \right\} \]

\[(4.28)\]

and

\[ z_1 = y + ix, \quad z_2 = y - ix, \]

\[ H_{PQ} = \sqrt{(h^2 - p^2)(h^2 - q^2)}, \quad U_{PQ} = \sqrt{(u^2 + p^2)(u^2 + q^2)} \]

\[ Z_{1PQ} = \sqrt{(z_1^2 - p^2)(z_1^2 - q^2)}, \quad Z_{2PQ} = \sqrt{(z_2^2 - p^2)(z_2^2 - q^2)}. \]

Here \( i = \sqrt{-1} \), and \( F \) and \( \Pi \) are the incomplete elliptic integrals of the first and third kinds, respectively. Equation (4.26) describes \( A(x, y) \) as a function of the parameters \( h, p, \) and \( q, \) and it has the property that \( A(x, 0) \) along the base boundary is invariant with respect to the values of \( h, p, \) and \( q \) (i.e., the base boundary is line tied). The functions \( A_1(x, y) \) in (4.27), \( A_2(x, y) \) in (4.28), and therefore \( A(x, y) \) in (4.26) are real, even though they explicitly contain complex numbers \( z_1 \) and \( z_2 \). Equations (4.27) and (4.28) are the simplest forms we can achieve, but they still contain some integrals which cannot be evaluated in closed forms. However, these integrals can be integrated numerically. Figure 4-2 plots contours of \( A(x, y) \) given by (4.26) with \( h = 25, p = 6, \) and \( q = 15 \) in a region of \(-20 \leq x \leq 20 \) and \( 0 \leq y \leq 40 \) in the unit of \( d. \)

Due to the lack of a closed form for the integrals for the dipole boundary condition, we will not determine the full dynamic evolution of this system following the loss of equilibrium. Instead, we focus on two other aspects: the first is whether a configuration with a detached current sheet such as that in Figure 4-2 can be in equilibrium, and the second is a determination of the maximum energy that can be released by the ideal-MHD jump in
Figure 4-2: Contours of the solution for the flux function $A(x, y)$ given by (4.26). For this configuration $h = 25$, $p = 6$, and $q = 15$ in units of $d$. "+" indicates the center of the flux rope.
this model.

### 4.4.2 Energetics

Substituting equations (4.24) and (4.25) into (4.23), and combining with (4.14) and (3.10), yields the total force per unit length acting on the flux rope:

\[
F = \frac{\hat{y}^2}{d} \left( \frac{I_0}{c} \right)^2 \left( \frac{2}{\pi} \right)^2 \frac{h^2 - p^2)(h^2 - q^2)}{h^2(p^2 - 1)^2(q^2 - 1)^2} \left\{ \frac{p^2(q^2 - 1)}{p^2 - 1} K(k') - \frac{q^2}{h^2 - 1} E(k') \right\} \cdot \left\{ \frac{q^2}{h^2 - 1} [K(k') - E(k')] - \frac{q^2 - 1}{h^2 - 1} K(k') \right\}
\]

\[
+ \frac{3 - 2q^2 + p^2(q^2 - 2) + h^2(p^2q^2 - 1)}{(h^2 - 1)^2(1 - p^2)} [K(k') - \Pi(1 - p^2, k')]\]

\[
- \frac{2h^2(p^2 - 1)(q^2 - 1)}{(h^2 - p^2)(h^2 - 1)} \left[ K(k') - \frac{p^2}{h^2} \Pi(1 - p^2, k') \right] - \frac{1}{4h^2} \left( \frac{h^2 + p^2}{h^2 - p^2} + \frac{2h^2}{h^2 - q^2} \right)
\]

\[
\cdot \left[ \frac{p^2(q^2 - 1)}{p^2 - 1} \Pi(1 - p^2, k') - \frac{p^2(q^2 - 1)}{p^2 - 1} K(k') - \frac{q^2}{h^2 - 1} E(k') \right]. \tag{4.29}
\]

We presume that fast reconnection is still prohibited as the loss of equilibrium occurs, so the tension force associated with the current sheet prevents the flux rope from escaping. Thus, there is an upper equilibrium location that is determined by setting \( F = 0 \) in equation (4.29).

As mentioned before, the catastrophe takes place within the Alfvén time scale, \( \tau_A \), and the magnetic flux, \( \phi \), is transported from below the photosphere to the corona within the photospheric motion time scale, \( \tau_p \). So, no magnetic flux enters the corona during the ideal-MHD jump because of (4.1), and two values of flux are conserved during the jump. They are the flux between the photosphere and the current sheet, \( \phi_1 \), and the flux between the photosphere and the surface of the flux rope, \( \phi_2 \), respectively. From (4.26), we have

\[
\phi_1 = A(0, p) - A(0, 0)
\]
\[
\phi_2 = A(0, h - r_0) - A(0, 0).
\]

The first conservation leads to

\[
\frac{1}{q} \left( \frac{P_0 K \left( \frac{p}{q} \right)}{h} - \frac{J}{\sqrt{(h^2 - p^2)(h^2 - p^2)}} \Pi \left( \frac{p^2}{h^2}, \frac{p}{q} \right) \right) + \frac{2K(p/q)}{\pi q(q^2 - 1)} \left\{ K(k') + q^2 [E(k') - 2K(k')] - \frac{q^2(p^2 - 2) + 1}{1 - p^2} [K(k') - p^2 \Pi(1 - p^2, k')] \right\} - \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{u^2 du}{(u^2 + 1)^2} \frac{\Pi \left[ \frac{p^2}{q^2} \left( \frac{u^2 + q^2}{u^2 + p^2} \right), \frac{p}{q} \right]}{\sqrt{(u^2 + p^2)(u^2 + q^2)}} = \frac{2\hbar c}{(h\hbar + y_0c)} \ln \left( \frac{\hbar c + y_0c}{\hbar c - y_0c} \right) - \frac{y_0c}{y_0c + 1}, \tag{4.30}
\]

and the second conservation gives

\[
\frac{2P_0}{q} \left[ \sin^{-1} \left( \frac{q}{h} \right), \frac{p}{q} \right] + J \ln \left( \frac{2hJ}{r_{00}} \right) - \frac{2J}{h\hbar} \sqrt{(h^2 - p^2)(h^2 - q^2)} \Pi \left[ \sin^{-1} \left( \frac{q}{h} \right), \frac{p^2}{h^2}, \frac{p}{q} \right] + \frac{4(q^2 - p^2)}{\pi q} \int_{-\infty}^{\infty} \frac{u^2 du}{(u^2 + 1)^2} \frac{\Pi \left[ \sin^{-1} \left( \frac{q}{h} \right), \frac{u^2 + q^2}{u^2 + p^2}, \frac{p}{q} \right]}{\sqrt{(u^2 + p^2)(u^2 + q^2)}} - \frac{4J}{\pi q^2 (p^2 - 1)^2} \left\{ q^2(1 - p^2) E(k') \right\},
\]

\[
+ \frac{4\sqrt{(h^2 - q^2)/(h^2 - p^2)}}{\pi \hbar q(h^2 - 1)[h^2(q^2 - 1) - q^2(p^2 - 1)]} \left\{ \frac{h^2(h^2 - p^2)}{p^2(1 - p^2, k')} + \frac{q^2(p^2 - 2) + 1}{p^2 - 1} \right\} + K(k') + h^2(q^2 - 1) - q^2(p^2 - 1) \left\{ h^2 K(k') - p^2 \Pi(1 - p^2/h^2, k') \right\} + q^2(h^2 - 1)(h^2 - p^2) \left\{ K(k') - \frac{p^2(h^2 - q^2)}{q^2(h^2 - p^2)} \Theta \left( \frac{h^2 q^2 - p^2}{q^2 h^2 - p^2}, k' \right) \right\} = J_c \ln \left( \frac{2\hbar c k}{r_{00}} \right) + \frac{2}{\hbar c + 1}, \tag{4.31}
\]

where the right hand sides of these equations are the values of \( \phi_1 \) and \( \phi_2 \) of the system at the critical (nose) point (Figure 4-1), and \( y_0c \) is the corresponding value of the neutral.
point's height, $y_0$, given by

$$y_0 = \frac{h(h+3)(h-1)}{(h+1)\sqrt{(h+1)(5h-3)+4h}}. \quad (4.32)$$

For a given $r_{oo}$, $\mathbf{F} = 0$, together with equations (4.30), (4.31), and (4.32), determines the values of $h$, $p$, and $q$ of the upper equilibrium position, $h_u$, $p_u$, and $q_u$, simultaneously. With these values, we can calculate the energy (per unit length) released by the ideal-MHD jump:

$$\Delta W = W_c - W_u = -\int_{h_c}^{h_u} F(h)du, \quad (4.33)$$

where $W_c$ and $W_u$ are the total energy per unit length stored in the system as the flux rope is located at the critical point and the upper equilibrium position, respectively. $F(h)$ takes the magnitude of $\mathbf{F}$ in (4.29), $p$ and $q$ are related to $h$ via (4.30) and (4.31). Therefore, a general solution for the equilibria of the configuration with a detached current sheet involves solving three coupled transcendental equations. Unfortunately, there is no closed form for the solutions due to the use of dipole boundary condition.

### 4.4.3 Asymptotic Solution for Small Radius

Xu [1992] investigated the situation of a small current sheet via a simple expansion of these equations. We now study the asymptotic solutions for small radius ($r_{oo} \to 0$) and look for an asymptotic relationship between the energy released and the radius $r_{oo}$.

As $r_{oo}$ goes to zero, the upper equilibrium height, $h_u$, approaches infinity, and so does $q_u$. In this case, the above equations simplify, and we can get some analytical results. As
\( r_{oo} \to 0 \), equations (4.19) and (4.32) reduce to

\[
y_{oc} = \frac{1}{2 \ln(2/r_{oo})} - \frac{3}{8 \ln^2(2/r_{oo})} + \cdots, \tag{4.34}
\]

and equation (4.30) becomes (without making any prescribed assumptions on \( p_u \) other than \( p_u/h_u \ll 1 \) and \( p_u/q_u \ll 1 \))

\[
p_u^2 \left[ \ln \left( \frac{2}{p-u} \right) - 1 \right] = \frac{1}{4 \ln^2(2/r_{oo})} - \frac{3}{8 \ln^3(2/r_{oo})},
\]

or

\[
p_u \sqrt{\ln \left( \frac{2}{p_u} \right) - 1} = \frac{1}{2 \ln 2/r_{oo}} - \frac{3}{8 \ln^2(2/r_{oo})}. \tag{4.35}
\]

Note that \( p_u \to 0 \) as \( r_{oo} \to 0 \), and we are back to the case of an attached current sheet. By means of numerical calculation one can show that the left hand side of equation (4.30) is nearly independent of \( h \) and \( q \) when both of them become large.

Simplifying (4.25) leads to

\[
J = \frac{2 \sqrt{h^2 - q^2}}{\pi q} \left[ 1 - p^2 \ln \left( \frac{2}{p} \right) \right] = \frac{2 \sqrt{h^2 - q^2}}{\pi q}, \tag{4.36}
\]

and equation (4.29) gives

\[
F(h) = \frac{2 l_0^2}{dc^2} \frac{J}{4h} \left[ J + \frac{2h^2 J}{h^2 - q^2} - \frac{8 \sqrt{h^2 - q^2}}{\pi q} \right] = \frac{2 l_0^2}{dc^2} \frac{J}{2h} \left[ -\frac{1}{2} + \frac{q^2}{h^2 - q^2} \right], \tag{4.37}
\]
where (4.36) has been. Simplifying flux conservation (4.31) gives

$$J \ln \left( \frac{2hJ}{r_{00}} \right) + 4 \left( \frac{Jq}{\sqrt{h^2 - q^2}} - \frac{1}{\pi} \right) \sin^{-1} \left( \frac{q}{h} \right) = \ln \left( \frac{2}{r_{00}} \right) + 1$$

(4.38)

for very large \( h \) and \( q \), and very small \( p \) and \( r_{00} \). Setting \( F(h) \) in (4.37) equal to zero and using (4.36) to relate \( q_u \) to \( h_u \), gives the upper equilibrium sheet height \( q_u \) and sheet current \( J_u \) as

$$q_u = h_u / \sqrt{3}, \quad J_u = 2\sqrt{2} / \pi.$$  

(4.39)  

(4.40)

Substituting these values into (4.38) relates the upper equilibrium height, \( h_u \), to the flux rope radius, \( r_{00} \):

$$h_u = \beta r_{00}^{-\alpha}.$$  

(4.41)

where

$$\alpha = \frac{\pi \sqrt{2}}{4},$$

$$\beta = 2^{\alpha - 3/2} \pi \exp \left\{ \sqrt{2} \left[ \frac{\pi}{4} - \sin^{-1} \left( \frac{1}{\sqrt{3}} \right) \right] \right\}.$$  

(4.42)

The above expressions, except for \( \beta \), are the same as those obtained for the case of the attached current sheet. Since \( \beta \) is actually a higher order correction in the expressions around the small parameter \(-1/\ln r_{00}\), the two cases are the same to lowest order in the expressions.

To calculate the energy, we use the method suggested by Isenberg et al. [1993] to
calculate both $W_u$ and $W_c$. First, we take the total derivative with respect to $h$ of the both
sides of (4.38), keeping in mind that the right hand side is always a constant for a given
$r_{00}$. Multiplying the resulting equation by $I_0 J/cd$ and subtracting equation (4.37) from
that product yields

$$F(h) = \frac{I_0^2}{dc^2} \left\{ \frac{d}{dh} \left\{ J^2 \left[ \ln \left( \frac{2h J}{r_{00}} \right) + \frac{1}{2} \right] \right\} + \frac{16}{\pi^2} \frac{d}{dh} \left[ \ln \left( \frac{q}{h} \right) \right] + \frac{8}{\pi^2 h} \right\}, \quad (4.43)$$

which changes (4.33) to

$$\Delta W = W_c - \frac{I_0^3}{c^2} \left\{ J_u^2 \left[ \ln \left( \frac{2h J_u}{r_{00}} \right) + \frac{1}{2} \right] + \frac{16}{\pi^2} \ln \left( \frac{q_u}{h_u} \right) + \frac{8}{\pi^2} \ln h_u \right\}. \quad (4.44)$$

where

$$W_c = \frac{I_0^3}{c^2} \left\{ J_c^2 \left[ \ln \left( \frac{2h_c J_c}{r_{00}} \right) + \frac{1}{2} \right] \right\}. \quad (4.45)$$

Equation (4.36) was used when we deduce (4.43). Plotting $\Delta W/W_c$ against $r_{00}$ indicates
that $\Delta W/W_c$ decreases with $r_{00}$, and the maximum of $\Delta W/W_c$ occurs as $r_{00} \to 0$ (Figure
4-3). As $r_{00}$ goes to zero, we have

$$\lim_{r_{00} \to 0} \frac{\Delta W}{W_c} = \lim_{r_{00} \to 0} \frac{W_c - W_u}{W_c} = \left( \frac{4\sqrt{2}}{\pi} - 1 - \frac{8}{\pi^2} \right), \quad (4.46)$$

which is the same as obtained for the case with an attached current sheet. Noted that $r_{00}$
must be extremely small for the asymptotic solution to hold. This is because $r_{00}$ appears
in the argument of a logarithm, and its resultant effect varies extremely slowly.
4.5 Discussions of Results

The difference between the case with an attached current sheet and that with a detached current sheet depends on whether a current sheet forms before or after the catastrophe. At the limit of $r_{00} \to 0$, the development of the current sheet and the catastrophe in the model with a dipole boundary condition occur simultaneously. So, the two cases should give the same output in the zero-radius limit. Note that although invoking reconnection ($\tau_R \ll \tau_p$) changes the topology of the configuration prior to the loss of equilibrium, the value of X-line height, $y_{oc}$, is quite small for $r_{00}$ within reasonable range. For example, $y_{oc} \approx 0.08$ when $r_{00} = 0.01$. Thus, whether a current sheet exists prior to the catastrophe does not significantly affect the dependence of $\Delta W/W_c$ on the radius $r_{00}$. Later on, we will find in Chapter 5 that when we consider the two-point source boundary condition, whether reconnection occurs or not prior to the loss of equilibrium never becomes an issue because
the two-point source configuration never produces X-line prior to the loss of equilibrium for any value of \( r_{00} \).

What we have discovered by analyzing the dipole boundary condition is that allowing reconnection to occur prior to the eruption eliminates the tension force associated with the current sheet attached to the base. This makes it much easier for the system to lose equilibrium and in the limit that the reconnection is rapid relative to the photospheric motion time scale (i.e. \( \tau_R \ll \tau_p \)), no critical radius occurs at all. Thus, under the plausible assumption that \( \tau_R \ll \tau_p \), the absence of catastrophe found by Forbes and Isenberg [1991] for reasonable values of \( r_{00} \) no longer holds. However, as we will discuss in Chapter 5, the dipole boundary condition still has the limitation of being rather feeble with respect to the magnetic energy it releases during the ideal-MHD jump. In fact, almost any other boundary condition (except for that corresponding to a line current below the surface) yields a more energetic ideal-MHD catastrophe than the dipole boundary condition.
Chapter 5

A Self-Consistent Model for Coupling of the CME Driving Force and the Reconnection Process

In this chapter, we investigate how magnetic reconnection affects the acceleration of eruptions and how the acceleration in turn affects the reconnection process. However, instead of using a model with a dipole boundary condition, we use the model with the two-point source boundary condition. This boundary condition provides the largest possible energy release during ideal-MHD catastrophe for a photospheric background field which is simply connected [Forbes et al. 1994]. As we have already seen, in the complete absence of reconnection the tension force associated with the current sheet is always strong enough to prevent the flux rope from escaping from the Sun, and this is always true whether or not an X-line or current sheet exists before the eruption. However, as we will show in this chapter, our new results imply that even a fairly small reconnection rate is sufficient to allow the flux rope to escape. Specifically, for a coronal density model that decreases exponentially with height, we find that the average Alfvén Mach number $M_A$ for the inflow into the reconnection site can be as small as $M_A = 0.005$ and still be fast enough to give a plausible eruption. The best fit to observations is obtained by assuming an inflow rate on the order of $M_A \approx 0.1$. With this value the energy output matches the temporal behavior inferred from
CME and long duration events (LDEs). (LDEs are CMEs accompanied by flares.) Our model also suggests an explanation for the peculiar motion of giant X-ray arches reported by Švestka et al. [1995, 1997]. X-ray arches are the large loops associated with CMEs which are similar in form to "post"-flare loops, but they have an upward motion that is often different. Instead of continually slowing with time, the arches move upward at a rate that remains nearly constant or may even increase with time. Here we show how the difference can be explained by reversal of the gradient of the coronal Alfvén speed with height.

5.1 Introduction

Models that attempt to explain solar eruptive flares must account for the rapid ejection of magnetic flux and plasma into interplanetary space. To explain the rapid time scale of the ejection, many authors have considered the idea that the primary mechanism for driving an eruption is a catastrophic loss of MHD equilibrium or stability (e.g., Van Tend and Kuperus [1978]; Van Tend [1979]; Birn and Schindler [1981]; Kaastra [1985]; Molodenskii and Filippov [1987]; Démoulin and Priest [1988]; Moore [1988]; Amari and Aly [1989]; Martens and Kuin [1989]; Anzer and Ballester [1990]; Priest and Forbes [1990]; Wolfson and Low [1992]; Low [1990, 1996]). Because mechanisms of this type operate on the Alfvén time scale, they can easily account for the rapid time scale of the eruption onset. However, by itself, a loss of equilibrium is not sufficient to explain the eruptive process. At some point in time, magnetic reconnection must occur to allow the solar magnetic field to relax to its pre-eruptive state. In fact, it may not be possible for mass to escape into interplanetary space unless magnetic reconnection occurs earlier. This is the situation for the two-dimensional model we consider in this part of work that is based on the earlier study by Forbes and Priest [1995]. They developed a model that contains a force-free flux rope that loses equilibrium when the two
photospheric sources of the coronal magnetic field are moved together as shown in Figure 3-7. When equilibrium is lost, the flux rope jumps to an equilibrium position at a higher altitude and a vertical current sheet is created. In order for this process to evolve into an eruption, reconnection must occur in the current sheet at a sufficiently rapid rate to allow the flux rope to escape smoothly (i.e., without substantial deceleration).

In next section we present the model description, and in section 5.3 we compute the solution for the vector potential of the magnetic field. In section 5.4 we derive the differential equations and present the initial conditions which govern the system’s evolution. In section 5.5 we examine the evolution of the configuration in our model following the loss of equilibrium. In section 5.6 we compute the magnetic energy released as a function of time. In section 5.7 we investigate the variations of the length of current sheet as well as the electric field there, and finally, we summarize our results in section 5.8.

5.2 Description of the Model

The question we address here is: How fast must the reconnection in the current sheet be in order for the flux rope to escape smoothly? In our model we express the rate of reconnection in terms of the Alfvén Mach number, $M_A$, at the midpoint of the current sheet, and we require this number to be less than unity. Although there is still much dispute about how fast reconnection can occur in the corona, there is a strong consensus that the Alfvén Mach number of the plasma flowing into the reconnection site cannot exceed unity.

For this model with two point sources located at $(\pm \lambda, 0)$, there are two T-type neutral points on the surface (at $z = \pm 2.2$ and $z = \pm 0.56$ in Figures 3-7b and 3-7c, respectively). However, no reconnection occurs at them since the flux function $A$ is invariant at the boundary due to the line-tying there. During the slow evolution of the boundary conditions
Figure 5-1: Diagram of the magnetic configuration for the two-point source model. Labels indicate the mathematical notation used in the text.
the two point sources approach one another and so do the two T-type neutral points. Eventually, a critical point is reached, and the flux rope jumps upward owing to a loss of equilibrium (Figure 3-7a). During the jump the two T-type neutral points merge at the origin to form an X-type neutral point out of which grows a current sheet. In the case of no reconnection, the current sheet remains attached to the surface (Figure 3-7d), but when reconnection occurs, it becomes detached (Figures 3-4, 4-2 and 5-1).

When dealing with the dynamics of the eruption, we can still use the six assumptions mentioned in section 4.2.1, and the basic equations (3.4), (3.5), (3.6), and (3.7), but for the boundary condition, we now use (3.15) which corresponds to a two-point source background field with point magnetic charges at \((x = \pm \lambda, \ y = 0)\). Forbes et al. [1994] and Forbes and Priest [1995] showed that an ideal-MHD jump in this structure releases \(~8\% of the magnetic energy stored in the system.

### 5.3 Solution for the Flux Function

In the absence of the flux rope and the current sheet, the field \(A_{ph}\), due to the photospheric source alone, is

\[
A_{ph}(x, y) = \frac{A_0}{\pi} \left[ \tan^{-1} \left( \frac{x + \lambda}{y} \right) - \tan^{-1} \left( \frac{x - \lambda}{y} \right) \right]. \quad (5.1)
\]

When the flux rope and the current sheet are included, two more contributions add to the expression for the vector potential \(A(x, y)\). We have upon substitution of (3.15) into (4.8):

\[
A(x, y) = \frac{I_0}{c} \left( J \ln \left( \frac{x^2 + (y + h)^2}{x^2 + (y - h)^2} \right) + \tan^{-1} \left( \frac{x + \lambda}{y} \right) - \tan^{-1} \left( \frac{x - \lambda}{y} \right) \right)
\]
\[ + \int_{p}^{q} \ln \left[ \frac{x^2 + (y + v)^2}{x^2 + (y - v)^2} \right] K_s(v) dv \] 

(5.2)

where \( A_0 \) has been normalized to \( I_0 \) such that \( A_0/\pi = I_0/c \). Equations (4.9) and (4.10) therefore become

\[ B_x(0, y) = \frac{I_0}{c} \left\{ \frac{4Jh}{h^2 - y^2} - \frac{2\lambda}{y^2 + \lambda^2} + \int_{p}^{q} \frac{4vK_s(v) dv}{v^2 - y^2} \right\} . \] 

(5.3)

and

\[ \int_{p}^{q} \frac{4vK_s(v) dv}{v^2 - y^2} = \frac{2\lambda}{y^2 + \lambda^2} - \frac{4Jh}{h^2 - y^2} \quad p \leq y \leq q. \] 

(5.4)

respectively. Compared with (4.13), equation (5.4) has a much simpler solution:

\[ K_s(y) = \frac{2Jh + \lambda + P_0}{\pi \sqrt{(y^2 - p^2)(q^2 - y^2)}} - \frac{2Jh \sqrt{(h^2 - p^2)(h^2 - q^2)}}{\pi \sqrt{(y^2 - p^2)(q^2 - y^2)(h^2 - y^2)}} - \frac{\lambda \sqrt{(\lambda^2 + p^2)(\lambda^2 + q^2)}}{\pi \sqrt{(y^2 - p^2)(q^2 - y^2)(\lambda^2 + y^2)}} \quad (\text{for } p \leq y \leq q), \] 

(5.5)

where \( P_0 \), as before, is a constant to be determined\(^1\).

Substituting (5.5) into (5.3) and completing the integration, we have (see Xu [1992] for some useful integral results)

\[ B_x(0, y) = \frac{2I_0}{c} \left\{ \frac{2Jh + \lambda + P_0}{\sqrt{(y^2 - p^2)(y^2 - q^2)}} + \frac{2Jh \sqrt{(h^2 - p^2)(h^2 - q^2)}}{\sqrt{(y^2 - p^2)(y^2 - q^2)(h^2 - y^2)}} \right\} \cdot \left\{ \begin{array}{ll} 1 & y < p \\ 0 & p \leq y \leq q \\ -1 & y > q. \end{array} \right. \] 

(5.6)

\(^{1}\)Note: one may find some typo in equation (16) of Lín and Forbes [2000], comparing it to (5.5).
Equation (5.6)\(^2\) indicates that \(B_s(0, y)\) has two singularities at the two tips of the current sheet, \((0, p)\) and \((0, q)\), unless the following equalities are met:

\[
2Jh + \lambda + P_0 - \frac{2Jh\sqrt{(h^2 - p^2)(h^2 - q^2)}}{h^2 - p^2} - \frac{\lambda\sqrt{(\lambda^2 + p^2)(\lambda^2 + q^2)}}{\lambda^2 + p^2} = 0, \\
2Jh + \lambda + P_0 - \frac{2Jh\sqrt{(h^2 - p^2)(h^2 - q^2)}}{h^2 - q^2} - \frac{\lambda\sqrt{(\lambda^2 + p^2)(\lambda^2 + q^2)}}{\lambda^2 + q^2} = 0,
\]

which lead to

\[
J = \frac{\lambda}{2h\sqrt{(\lambda^2 + p^2)(\lambda^2 + q^2)}} \sqrt{(h^2 - p^2)(h^2 - q^2)},
\]

\[
P_0 = \frac{\lambda\sqrt{(\lambda^2 + p^2) - (\lambda^2 + q^2)}}{\sqrt{\lambda^2 + p^2} + \sqrt{\lambda^2 + q^2}} \left( \sqrt{\frac{h^2 - p^2}{\lambda^2 + p^2}} - \sqrt{\frac{h^2 - q^2}{\lambda^2 + q^2}} \right),
\]

and thus

\[
2Jh + \lambda + P_0 = \frac{\lambda(h^2 + \lambda^2)}{\sqrt{(\lambda^2 + p^2)(\lambda^2 + q^2)}}.
\]

These conditions force the field to have a configuration with a Y-type neutral point at each tip of the current sheet as shown in Figure 5-1.

Substituting (5.7) through (5.9) into (5.5), and then into (5.2), we finally get, after some rather involved algebra, the explicit expression for \(A(x, y)\):

\[
A(x, y) = \frac{I_0}{c} \begin{cases} 
A_1(x, y) & x^2 + y^2 \geq p^2 \\
A_2(x, y) & x^2 + y^2 < p^2,
\end{cases}
\]

\(^2\)Note: one may find some typo in equation (17) of Lin and Forbes [2000], comparing it to (5.6).
where

\[
A_1 = J \ln \left( \frac{hZ_1 PQ + z_1 H_{PQ}}{hZ_1 PQ - z_1 H_{PQ}} \cdot \frac{hZ_2 PQ + z_2 H_{PQ}}{hZ_2 PQ - z_2 H_{PQ}} \right) \\
+ \frac{\lambda(h^2 + \lambda^2)}{qL_{PQ}} \left\{ F \left[ \sin^{-1} \left( \frac{q}{z_1} \frac{p}{q} \right), \frac{p}{q} \right] + F \left[ \sin^{-1} \left( \frac{q}{z_2} \frac{p}{q} \right), \frac{p}{q} \right] \right\} \\
- \frac{2J}{h} H_{PQ} \left\{ \Pi \left[ \sin^{-1} \left( \frac{z_1}{p} \frac{p^2}{h^2} \frac{p}{q} \right), \frac{p^2}{h^2}, \frac{p}{q} \right] + \Pi \left[ \sin^{-1} \left( \frac{z_2}{p} \frac{p^2}{h^2}, \frac{p}{q} \right) \right] \right\} \\
+ \frac{\lambda(q^2 - p^2)}{qL_{PQ}} \left\{ \Pi \left[ \sin^{-1} \left( \frac{z_1}{p} \frac{p^2}{q^2} \lambda^2 + p^2 \frac{p}{q} \right), \frac{p^2}{q^2}, \frac{p}{q} \right] + \Pi \left[ \sin^{-1} \left( \frac{z_2}{p} \frac{p^2}{q^2}, \lambda^2 + p^2, \frac{p}{q} \right) \right] \right\} \\
- \frac{\lambda}{q} \sqrt{\frac{\lambda^2 + q^2}{\lambda^2 + p^2}} \left\{ F \left[ \sin^{-1} \left( \frac{z_1}{p} \frac{p^2}{q^2} \lambda^2 + p^2 \frac{p}{q} \right), \frac{p}{q} \right] + F \left[ \sin^{-1} \left( \frac{z_2}{p} \frac{p^2}{q^2}, \lambda^2 + p^2, \frac{p}{q} \right) \right] \right\} \\
+ \tan^{-1} \left\{ \frac{z_1}{z_2} \sqrt{\frac{(\lambda^2 + p^2)(z_1^2 - p^2)}{(\lambda^2 + q^2)(z_1^2 - q^2)}} \right\} + \tan^{-1} \left\{ \frac{z_2}{z_2} \sqrt{\frac{(\lambda^2 + p^2)(z_2^2 - p^2)}{(\lambda^2 + q^2)(z_2^2 - q^2)}} \right\}, \quad (5.11)
\]

\[
A_2 = \pi + \frac{\lambda(h^2 + \lambda^2)}{qL_{PQ}} \left\{ F \left[ \sin^{-1} \left( \frac{z_1}{p} \frac{p^2}{q^2} \lambda^2 + p^2 \frac{p}{q} \right), \frac{p}{q} \right] + F \left[ \sin^{-1} \left( \frac{z_2}{p} \frac{p^2}{q^2}, \lambda^2 + p^2, \frac{p}{q} \right) \right] \right\} \\
- \frac{2J}{h} H_{PQ} \left\{ \Pi \left[ \sin^{-1} \left( \frac{z_1}{p} \frac{p^2}{q^2} \lambda^2 + p^2 \frac{p}{q} \right), \frac{p^2}{q^2}, \frac{p}{q} \right] + \Pi \left[ \sin^{-1} \left( \frac{z_2}{p} \frac{p^2}{q^2}, \lambda^2 + p^2, \frac{p}{q} \right) \right] \right\} \\
+ \frac{\lambda(q^2 - p^2)}{qL_{PQ}} \left\{ \Pi \left[ \sin^{-1} \left( \frac{z_1}{p} \frac{p^2}{q^2} \lambda^2 + p^2 \frac{p}{q} \right), \frac{p^2}{q^2}, \frac{p}{q} \right] + \Pi \left[ \sin^{-1} \left( \frac{z_2}{p} \frac{p^2}{q^2}, \lambda^2 + p^2, \frac{p}{q} \right) \right] \right\} \\
- \frac{\lambda}{q} \sqrt{\frac{\lambda^2 + q^2}{\lambda^2 + p^2}} \left\{ F \left[ \sin^{-1} \left( \frac{z_1}{p} \frac{p^2}{q^2} \lambda^2 + p^2 \frac{p}{q} \right), \frac{p}{q} \right] + F \left[ \sin^{-1} \left( \frac{z_2}{p} \frac{p^2}{q^2}, \lambda^2 + p^2, \frac{p}{q} \right) \right] \right\} \\
- \tan^{-1} \left\{ \frac{z_1}{z_2} \sqrt{\frac{(\lambda^2 + p^2)(z_1^2 - p^2)}{(\lambda^2 + q^2)(z_1^2 - q^2)}} \right\} - \tan^{-1} \left\{ \frac{z_2}{z_2} \sqrt{\frac{(\lambda^2 + p^2)(z_2^2 - p^2)}{(\lambda^2 + q^2)(z_2^2 - q^2)}} \right\}, \quad (5.12)
\]

and

\[ z_1 = y + ix, \quad z_2 = y - ix, \]

\[ H_{PQ} = \sqrt{(h^2 - p^2)(h^2 - q^2)}, \quad L_{PQ} = \sqrt{(\lambda^2 + p^2)(\lambda^2 + q^2)}, \]

\[ Z_{1PQ} = \sqrt{(z_1^2 - p^2)(z_1^2 - q^2)}, \quad Z_{2PQ} = \sqrt{(z_2^2 - p^2)(z_2^2 - q^2)}. \]

Here \( i = \sqrt{-1}, \) and \( F \) and \( \Pi \) are the incomplete elliptic integrals of the first and third kinds, respectively. Equation (5.10) describes \( A(x, y) \) as a function of the parameters \( \lambda, h, p, \) and \( q, \) and it has the property that \( A(x, 0) \) along the base boundary is invariant with
respect to the values of \( h, p, \) and \( q \) (i.e., the base boundary is line tied). The functions \( A_1(x, y) \) in (5.11), \( A_2(x, y) \) in (5.12), and therefore \( A(x, y) \) in (5.10) are real, even though they explicitly contain complex numbers \( z_1 \) and \( z_2 \). Obviously, equations (5.10), (5.11), and (5.12) have the same form as (4.26), (4.27), and (4.28) except for those integrals which are not in closed form. Figure 5-2 plots contours of \( A(x, y) \) given by (5.10) with \( \lambda = \lambda_c = 0.97 \) (the critical value of \( \lambda \) at which the catastrophic loss of equilibrium occurs), \( h = 25, \ p = 6, \) and \( q = 15 \) in the region \(-15 \leq x \leq 15 \) and \( 0 \leq y \leq 30 \) in units of \( 5 \times 10^4 \) km.

Comparing Figure 5-2 with Figure 3-4 indicates some similarity between them. A. Van Ballegooijen (private communication) has pointed to us that the magnetic configurations in Figure 5-2 can also be deduced directly from that in Figure 3-4 by splitting the single line-dipole source on the boundary surface into two line-monopole sources at \( x = -\lambda \) and \( x = \lambda \), respectively. In Appendix A, we show that this is indeed correct. This allows both the magnetic field \( B(z) \) and the complex flux function \( A(z) \) to be expressed in the compact form:

\[
B(z) = B_y + iB_x = \frac{2I_0i}{c} \frac{\lambda(h^2 + \lambda^2)\sqrt{(z^2 + p^2)(z^2 + q^2)}}{\sqrt{(\lambda^2 + p^2)(\lambda^2 + q^2)(z^2 - \lambda^2)(z^2 + h^2)}},
\]

and

\[
A(z) = \frac{2I_0}{c} \frac{1}{(q - p)\sqrt{(\lambda^2 + p^2)(\lambda^2 + q^2)}} \left( 2i\lambda(\lambda^2 + q^2) \right)^{-1} \cdot F \left\{ \sin^{-1} \left[ \frac{(p - q)(p - iz)}{(p + q)(p + iz)} \right], \frac{p + q}{p - q} \right\}
\]

\[3\text{Note: one may find some typo in equation (23) of Lin and Forbes [2000], comparing it to (5.12).}\]
Figure 5-2: Contours of the solution for the flux function $A(x, y)$ given by equation (5.10) for $\lambda = \lambda_c$, $h = 25$, $p = 6$, $q = 15$, and $J$ is determined by equation (5.7). All the lengths are in units of $5 \times 10^4$ km.
\[ + \frac{\lambda}{q} (p - q) (\lambda^2 + h^2) F \left[ i \sinh^{-1} \left( \frac{z}{p} \right), \frac{p}{q} \right] \]
\[ + \frac{\lambda}{h^2 q} (q - p) (h^2 - p^2) (h^2 - q^2) \Pi \left[ i \sinh^{-1} \left( \frac{z}{p} \right), \frac{p^2}{h^2}, \frac{p}{q} \right] \]
\[ - 2p (\lambda^2 + p^2) \Pi \left\{ \sin^{-1} \left[ \sqrt{\frac{(p - q)(p - iz)}{(p + q)(p + iz)}}, \frac{(p + q)(p - i\lambda)}{(p - q)(p + i\lambda)}, \frac{p + q}{p - q} \right] \right\} \]
\[ + 2p (\lambda^2 + p^2) \Pi \left\{ \sin^{-1} \left[ \sqrt{\frac{(p - q)(p - iz)}{(p + q)(p + iz)}}, \frac{(p + q)(p + i\lambda)}{(p - q)(p - i\lambda)}, \frac{p + q}{p - q} \right] \right\} , \quad (5.14) \]

where both \( B_x \) and \( B_y \) are real, \( z = x + iy \), and \( A(x, y) \) in (5.10) is related to \( A(z) \) in the way of \( A(x, y) = \Re[A(z)] \). So, taking the real of \( A(z) \) and using the same values for \( \lambda, h, \)
\( p, \) and \( q \) as before recovers Figure 5-2.

Comparing equation (5.13) with equation (3.1) shows that the two become equivalent when the following modifications in equation (3.1) are made: First, replace the single \( z^2 \) and the product \( pq \) in the denominator by \( (z^2 - \lambda^2) \) and \( \sqrt{(\lambda^2 + p^2)(\lambda^2 + q^2)} \), respectively; and second replace the single \( h^2 \) and \( m \) in the numerator by \( (h^2 + \lambda^2) \) and \( 2f_0 \lambda/c \), respectively.

Although equation (5.14) has a more compact form than its real counterpart described by equations (5.10) through (5.12), it is difficult to use to do further analysis and make plot of the field lines because of the multiple valuedness associated with the branch cuts of \( A(z) \). Finding the real part of \( A(z) \) given by (5.14) and evaluating those elliptic integrals with complex arguments, amplitudes, and parameters is a time-consuming and tedious process. Therefore, we use equations (5.10) through (5.12) as the basis for our analysis in the following sections.

### 5.4 Basic Equations for System’s Evolution

As reconnection occurs, the configuration shown in Figure 5-2 is generally not in equilibrium.

We are going to determine the velocity of flux rope, the length of current sheet, and the
electric field in the current sheet as functions of both the height of the flux rope, \(h\), and the time, \(t\). In a two-dimensional model the rate of reconnection is prescribed by the electric field \(E_z\) at the X-point or, in our case, in the current sheet. Using Faraday's equation, we can write

\[
E_z = -\frac{1}{c} \frac{\partial A_0^0}{\partial t} = -\frac{1}{c} \frac{\partial A_0^0}{\partial h} \dot{h},
\]

where \(A_0^0\) is the value of the flux function \(A(x, y)\) along the current sheet, and \(\dot{h} = dh/dt\).

Setting \(A_0^0 = A(0, p \leq y \leq q)\) gives

\[
A_0^0 = \frac{2I_0}{c} \frac{\lambda}{qLPQ} \left( (h^2 - q^2)K\left(\frac{p}{q}\right) + (q^2 - p^2)\Pi\left(\frac{\lambda^2 + p^2}{\lambda^2 + q^2},\frac{p}{q}\right) - \frac{H_{PQ}^2}{h^2} \Pi\left(\frac{p^2}{h^2},\frac{p}{q}\right) \right),
\]

where \(K\) and \(\Pi\) are the complete elliptic integrals of the first and third kinds, respectively. Note the difference between the arguments of the function \(\Pi\) in (5.16) and those in (5.11) and (5.12).

We now substitute \(A_0^0\) into (5.15), keeping in mind that \(p\) and \(q\) are functions of \(h\), and that \(\lambda\) is fixed. Making use of the following chain law

\[
\frac{\partial A_0^0}{\partial h} = \left( \frac{\partial A_0^0}{\partial p} \right)_{h,q} \frac{dp}{dh} + \left( \frac{\partial A_0^0}{\partial q} \right)_{h,p} \frac{dq}{dh} + \left( \frac{\partial A_0^0}{\partial h} \right)_{p,q},
\]

where the subscripts of partial derivatives mean keeping the corresponding parameters fixed, we have

\[
E_z = -\frac{\dot{h}}{c} (A_0^0 p' + A_0^0 q' + A_0^0),
\]

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where \( p' = dp/dh \) and \( q' = dq/dh \), and

\[
\begin{align*}
A_{op} & = \frac{\lambda p(h^2 + \lambda^2)(\lambda^2 + q^2)}{h^2 q[(\lambda^2 + p^2)(\lambda^2 + q^2)]^{3/2}} \left[ (h^2 - q^2) \Pi \left( \frac{p^2}{h^2}, \frac{p}{q} \right) - h^2 K \left( \frac{p}{q} \right) \right], \\
A_{og} & = \frac{\lambda (h^2 + \lambda^2)(\lambda^2 + p^2)}{h^2 [(\lambda^2 + p^2)(\lambda^2 + q^2)]^{3/2}} \left[ (h^2 - q^2) \Pi \left( \frac{p^2}{h^2}, \frac{p}{q} \right) - h^2 K \left( \frac{p}{q} \right) \right], \\
A_{oh} & = -\frac{\lambda}{h^3 q \sqrt{[(\lambda^2 + p^2)(\lambda^2 + q^2)]}} \left[ h^2 q^2 E \left( \frac{p}{q} \right) - h^2 (h^2 + q^2) K \left( \frac{p}{q} \right) \right] + (h^4 - p^2 q^2) \Pi \left( \frac{p^2}{h^2}, \frac{p}{q} \right).
\end{align*}
\]  

(5.19)

Here \( E \) is the complete elliptic integral of the second kind. Note that the length of the current sheet, namely, \((q - p)\), and the velocity of the flux rope \( \dot{h} \) depend on \( E_\gamma \), which describes the rate of the reconnection in the current sheet. To find the variations of \( \dot{h} \), \( p \), and \( q \) as functions of the flux rope height \( h \), we need two other equations: the frozen-flux condition at the surface of the flux rope and the dynamic equation describing the flux rope acceleration due to the forces acting upon it.

For the frozen flux condition we have

\[
\frac{2I_0}{c} A_R = A(0, h - r_0) = \text{const},
\]  

(5.20)

where \( r_0 \) is the radius of the flux rope.

Substituting (5.10), (5.11), and (5.12) into (5.20) and keeping in mind that \( h \gg r_0 \), we find

\[
\begin{align*}
A_R & = \frac{\lambda}{2h L_{PQ}} \ln \left[ \frac{\lambda H_{PQ}^{3/2}}{r_{00}(h^4 - p^2 q^2) L_{PQ}} \right] + \tan^{-1} \left( \frac{\lambda}{\dot{h}} \sqrt{\frac{(\lambda^2 + p^2)(h^2 - q^2)}{(\lambda^2 + q^2)(h^2 - p^2)}} \right) \\
& + \frac{\lambda}{q L_{PQ}} \left( (h^2 - q^2) E \left( \frac{q}{h}, \frac{p}{q} \right) + (q^2 - p^2) K \left( \frac{q}{h}, \frac{\lambda^2 + p^2}{\lambda^2 + q^2}, \frac{p}{q} \right) \right).
\end{align*}
\]
\[-\frac{H_{PQ}^2}{h^2} \Pi \left[ \sin^{-1} \left( \frac{q}{h}, \frac{p^2}{h^2}, \frac{p}{q} \right) \right] \]
\[= \frac{\pi}{4} \ln \left( \frac{2\lambda_0}{r_{00}} \right), \quad (5.21)\]

where \(H_{PQ}\), and \(L_{PQ}\) are as before. The parameters \(\lambda_0\), \(r_{00}\), and \(I_0\) in (5.21) are the values of \(\lambda\), \(r_0\), and \(I\), respectively, when the flux rope is located at the position on the equilibrium curve where the current in the flux rope has its maximum value (see Forbes and Isenberg [1991]; Isenberg et al. [1993]; and Lin et al. [1998]).

The radius of the flux rope, \(r_0\), is related to the current \(J\) by equation (4.17). The errors induced by using this approximation in the following calculations are not significant as long as \(r_{00} < 0.1\). After the catastrophic loss of mechanical equilibrium, the current intensity \(J\) decreases with \(h\) very slowly, from \(J \approx 1\) at \(h \approx 1\) (in the unit of \(5 \times 10^4\) km) to \(J \approx 0.6\) when \(h = 10^2\) (in the unit of \(5 \times 10^4\) km). At \(J = 1\), there is no error between \(r_0\) given by (4.17) and Parker’s solution; but at \(J = 0.6\), there is an error of \(\sim 10\%\). Since \(r_0\) only appears in the argument of a logarithm, the effect of this error on our final results reduces to less than \(6\%\), which is negligible considering our neglect of other important effects, such as shock waves, radiation losses, etc.

Taking the total derivatives about \(h\) on both sides of (5.21) gives

\[A_{Rp} p' + A_{Rq} q' + A_{Rh} = 0, \quad (5.22)\]

where

\[A_{Rp} = \left( \frac{\partial A_R}{\partial p} \right)_{h,q} = \frac{\lambda p(h^2 + \lambda^2)}{q(\lambda^2 + p^2)} \sqrt{\lambda^2 + p^2} \left( 1 - \frac{p^2}{h^2} \right) \Pi \left[ \sin^{-1} \left( \frac{q}{h}, \frac{p^2}{h^2}, \frac{p}{q} \right) \right] \]
\[ A_{Rq} = \left( \frac{\partial A_R}{\partial q} \right)_{h,p} \]

\[ = \frac{\lambda(h^2 + \lambda^2)}{(\lambda^2 + q^2)} \sqrt{\lambda^2 + q^2} \left( \frac{1 - p^2}{h^2} \right) \Pi \left[ \sin^{-1} \left( \frac{q}{h} \right) \right] \frac{p^2 - p}{h^2, q} \]

\[ - F \left[ \sin^{-1} \left( \frac{q}{h} \right), \frac{p}{q} \right] - \frac{q}{2h} \sqrt{\frac{h^2 - q^2}{h^2 - p^2}} \left\{ 1 + \ln \left[ \frac{\lambda H_{PQ}^2}{r_0 L_{PQ}(h^4 - p^2 q^2)} \right] \right\}, \]

\[ A_{Rh} = \left( \frac{\partial A_R}{\partial h} \right)_{p,q} \]

\[ = \frac{\lambda}{2h^2 L_{PQ} H_{PQ}} \left\{ \frac{2h^2 - 2(\lambda pq)^2}{h^2 - \lambda^2} - \frac{h^2(p^2 + q^2)(h^2 - \lambda^2)}{h^2 - \lambda^2} \right\} \]

\[ + (h^4 - p^2 q^2) \ln \left[ \frac{\lambda H_{PQ}^2}{r_0 L_{PQ}(h^4 - p^2 q^2)} \right] \}

\[ + \frac{\lambda}{hq L_{PQ}} \left\{ (h^2 + q^2) F \left[ \sin^{-1} \left( \frac{q}{h} \right) \right] - \frac{p^2}{h^2, q} \frac{p}{h^2} \left[ \sin^{-1} \left( \frac{q}{h} \right) \right] \frac{p}{q} \right\} \]

\[ - \left( h^2 - \frac{p^2 q^2}{h^2} \right) \Pi \left[ \sin^{-1} \left( \frac{q}{h} \right) \right] \frac{p^2}{h^2, q} \right\}. \] (5.23)

The dynamic equation relates the external force per unit length \( F \) acting on the flux rope to the acceleration of the flux rope \( \frac{d^2 h}{dt^2} \) by

\[ F = m \frac{d^2 h}{dt^2} \dot{y}, \] (5.24)

where \( m \) is the total mass per unit length of the flux rope, and

\[ F = \frac{I_0}{c} J B_{ext} \dot{y} \]

\[ = \left( \frac{I_0}{c} \right)^2 \frac{\lambda^2}{2h L_{PQ}^2} \left\{ \frac{H_{PQ}^2}{2h^2} - \frac{(\lambda^2 + p^2)(h^2 - q^2)}{\lambda^2 + h^2} - \frac{(\lambda^2 + q^2)(h^2 - p^2)}{\lambda^2 + h^2} \right\} \dot{y}. \] (5.25)

The field strength \( B_{ext} \) is the total external field evaluated at the center of the flux rope.

(For a more detailed description of the procedure, the readers should consult the previous...
chapter.) Substituting (5.25) into (5.24) and using

\[ \frac{d^2 h}{dt^2} = \frac{dh}{dt} \frac{d}{dt} \left( \frac{dh}{dt} \right) = \dot{h} \dot{h}', \]

we have

\[ m \dot{h} \dot{h}' = \left( \frac{I_0}{c} \right)^2 \frac{\lambda^2}{2hL_{\phi}^2} \left[ \frac{H_{\phi}^2}{2h^2} - \frac{(\lambda^2 + p^2)(h^2 - q^2)}{\lambda^2 + h^2} - \frac{(\lambda^2 + q^2)(h^2 - p^2)}{\lambda^2 + h^2} \right]. \quad (5.26) \]

Equations (5.18), (5.22), and (5.26) form a deterministic system for \( p, q, \) and \( \dot{h} \) as functions of \( h, \) with \( E_z \) as a function yet to be determined. The electric field \( E_z \) is a direct measure of the reconnection rate in a two-dimensional system since it describes the rate at which magnetic flux passes through the current sheet. The details of the reconnection process depend not only on the structure of the current sheet, but on the circumstances around the current sheet as well. These are in turn related to the local Alfvén velocity \( V_A. \)

As discussed earlier, we do not attempt in this work to solve the equations that determine the reconnection dynamics. Instead, we assume that the rate of reconnection is given either by observations or by a separate theoretical analysis. Many theories (e.g., Sweet-Parker, Petschek) use the Alfvén Mach number in the inflow of the reconnection region as a measure of the reconnection rate. We do the same here by using the Alfvén Mach number at the midpoint of the upstream boundary of the current sheet. This location corresponds to either \( (x_0 = 0^+, y_0 = (p + q)/2) \) or \( (x_0 = 0^-, y_0 = (p + q)/2) \), where \( 0^\pm \) indicates the region just outside of the discontinuity in \( B_y \) at either side of the current sheet. Thus

\[ M_A = \frac{|V_R|}{V_A}, \quad (5.27) \]
where
\[ V_R = -\frac{cE_z}{B_y(0^+, y_0)} = \frac{\partial A_0}{\partial t} \frac{1}{B_y(0^+, y_0)} \]  
(5.28)

is the velocity of the plasma flowing into the current sheet, and
\[ V_A = \frac{|B_y(0^+, y_0)|}{\sqrt{4\pi\rho(y_0)}} \]  
(5.29)
is the local Alfvén velocity at the same location. Combining (5.28) and (5.29) with (5.27), we obtain
\[ E_z = -\frac{1}{c} \frac{\partial A_0}{\partial t} = \frac{1}{c} \frac{M_A B_y^2(0^+, y_0)}{\sqrt{4\pi\rho(y_0)}}, \]  
(5.30)

where \( E_z \) is positive since \( A_0 \) decreases with time. In this chapter, we will call \( M_A \), as given by (5.27), the relative rate of reconnection, and \( E_z \), as given by (5.30), the absolute rate of reconnection. Although we assume that the relative rate \( M_A \) is constant, this does not mean that the absolute rate \( E_z \) is constant.

To determine \( E_z \) requires a model for the plasma density \( \rho(y) \), and we will consider here two different cases. The simplest one is that
\[ \rho(y) = \rho_0, \]  
(5.31)

where \( \rho_0 \) is a constant. This leads to an Alfvén velocity that decreases to zero at large distances as shown in Figure 5-3a. The other case is that
\[ \rho(y) = \rho_0 \exp\left(-\frac{y}{H_g}\right), \]

where \( H_g \), the gravitation height, is set to \( 10^5 \) km. Normalizing \( \rho \) to \( \rho_0 \), and both \( y \) and
Figure 5-3: Variation of (a) the background Alfvén speed as a function of height and (b) the local Alfvén at the midpoint of the current sheet as a function of flux-rope height for the case $M_A = 1$. Solid curves result from the isothermal atmosphere model and dashed curves from the uniform atmosphere model.
$H_g$ to $\lambda_0 \left(5 \times 10^4 \text{ km}\right)$, we have

$$\rho(y) = \exp \left(-\frac{y}{2}\right). \quad (5.32)$$

The second case leads to an Alfvén velocity that increases to infinity at large distances, and it is the more realistic of the two cases. However, we include the first case because it helps illustrate the effect of the density model upon our results. As we shall see, there is a significant difference in the long-term evolution of the two cases due to the difference in the variations of the Alfvén velocity with height.

Since there is no generally accepted theory for how fast reconnection occurs when it is driven by a loss of equilibrium, we will assume for simplicity that $M_A$ is a constant less than unity. In general, to calculate $M_A$ self-consistently from the equations, it is necessary to solve the time-dependent diffusive MHD equations in the vicinity of the current sheet. The key parameter that such a solution must provide is the thickness of the sheet since the convective velocity of the plasma flowing into the current sheet must be equal to the rate at which field lines diffuse through the current sheet. That is, $M_A \approx \eta/(IV_A)$, where $\eta$ is the magnetic diffusivity and $l$ is the thickness of the sheet.

Now if the evolution of the sheet is sufficiently slow, a steady-state theory like Sweet-Parker or Petschek can be used to express $l$ as a function of $M_A$ and $\eta$. Here “sufficiently slow” means that the time it takes for the current sheet to grow is long compared to the time it takes for an Alfvén wave to travel along it. Since the upper tip of the current sheet initially moves much more rapidly than the lower tip, the rate at which the sheet grows is approximately $\dot{q}$. Thus a quasi-steady treatment requires that $\dot{q} \ll V_A$. As we will see, this condition is not satisfied until 2 to 3 hours after the onset of the eruption, so a quasi-steady
treatment cannot be used except at relatively late times.

Even when a quasi-steady treatment can be justified, there is still the problem of which quasi-steady theory to use. If we assume that the Sweet-Parker theory is appropriate, then we have \( M_A = \eta / ([V_A(q - p)] \), where \( V_A \) is now the average Alfvén speed along the sheet. Alternatively, to apply Petschek's theory, we assume that the current sheet stretching from \( p \) to \( q \) actually consists of a much shorter current sheet with pairs of slow-mode shocks extending outward to \( p \) and \( q \) from each of its ends. For values of \( M_A \) in the range 0.01 to 0.1, the angle of separation between the shocks is only a few degrees, so the fact that the sheet is now bifurcated at its ends has little effect on the global field at large distances, except through the value of \( M_A \). Since Petschek's reconnection rate depends only on \( \ln \eta \), the result is almost the same as assuming that \( M_A = \text{const} \). Another possibility, which is also likely to give a nearly constant value of \( M_A \), is that the reconnection process is turbulent and therefore completely independent of \( \eta \). In the latter case the internal structure of the current sheet is quite complex and consists of many small current sheets and islands, but again, there is no effect on the global field at large distances, except through the value of \( M_A \).

With the additions of the density model (\( \rho = \rho_0 \) or \( \rho = \rho_0 \exp[-y/H_y] \)) and the reconnection model (\( M_A = \text{const} \)), we now have a complete system of equations for the evolution of the flux configuration with time. The system consists of (5.18), which is Faraday's equation applied at the current sheet, (5.22), which is the frozen flux condition at the surface of the flux rope, and (5.26), which is Newton's second law of motion applied to the flux rope. These three equations can be combined to give

\[
p' = \frac{\dot{A}_{0h}A_{Rq} - A_{Rh}A_{0q}}{A_{Rp}A_{0q} - A_{0p}A_{Rq}},
\]
\[ q' = \frac{ARh A_{0p} - \hat{A}_{0h} A_{Rp}}{AR_p A_{0q} - \hat{A}_{0p} A_{Rq}}, \]
\[ \hat{h}' = \left( \frac{I_0}{c} \right)^2 \frac{\lambda^2}{2m hh L_{PQ}^2} \frac{H_{PQ}^2}{2h^2} \left[ \frac{(\lambda^2 + p^2)(h^2 - q^2)}{\lambda^2 + h^2} - \frac{(\lambda^2 + q^2)(h^2 - p^2)}{\lambda^2 + h^2} \right], \]
(5.33)

where
\[ A_{0h} = \frac{c E_z}{h} + A_{0h} = \frac{M_A B^2 (0^+, y_0)}{h \sqrt{4\pi \rho(y_0)}} + A_{0h}. \]

As an illustrative example of our results, we use the following values:
\[ \lambda_0 = 5 \times 10^4 \text{ km}, \quad L = 10^5 \text{ km}, \quad r_{00} = 0.1\lambda_0, \]
\[ \rho_0 = 1.67 \times 10^{-14} \text{ gm/cm}^3, \quad I_0/(c \lambda_0) = 50 \text{ G}, \quad \hat{h}_0 = 1000 \text{ km/s}, \]

where \( L \) is a typical length for a flux rope and \( \rho_0 \) is the density at the base of corona.

Generally, the total mass of the ejecta is \( \sim 2.1 \times 10^{16} \text{ g} \), so the total mass per unit length of the flux rope is
\[ m = 2.1 \times 10^6 \text{ g/cm}. \]

These values are used for our following calculations in which all lengths are normalized to \( \lambda_0 \), velocities to \( \hat{h}_0 \), mass density to \( \rho_0 \), and current to \( I_0 \).

In deriving these equations we assume that there is no normal component of the field in the sheet, so that field lines merge all along the length of the sheet. In general, there will be a normal component, but it should be negligible once the length of the current sheet is much greater than its thickness. To illustrate this, let us consider the distribution of current
density within the sheet, namely,

\[ K_S(y) \approx \int_{-l/2}^{l/2} \left( \frac{\partial B_y}{\partial x} - \frac{\partial B_z}{\partial y} \right) dx \]

\[ = 2B_y(l/2, y) - l \frac{\partial B_z(l/2, y)}{\partial y}, \]

and let us also suppose that the Sweet-Parker model applies. Then, estimating \( \partial B_z/\partial y \)
by \( B_z/(q - p) \) and using the Sweet-Parker results that \( B_z \approx M_A B_y(l/2, (q - p)/2) \) and
\( l = M_A(q - p) \), we obtain

\[ K_S \approx 2(1 - M_A^2) B_y. \]

Thus the error in \( K_S \) is of second order in \( M_A \), or \( \sim 1\% \) for the value of \( M_A = 0.1 \) used in
most of our plots.

5.5 Evolution Following the Loss of Equilibrium

Before the flux rope loses equilibrium, it evolves slowly along the equilibrium curve shown
in Figure 3-7 until it reaches the critical point that occurs when the separation of the
photospheric sources reaches \( \lambda_c \). The equations that describe the equilibrium curve are
given by Forbes and Priest [1995], who showed that for a flux rope whose radius is 0.1 at
\( \lambda = 1 \) (the point on the curve where the flux rope current has its maximum value), the
separation at the critical point is \( \lambda = \lambda_c = 0.9695 \), the flux rope height is \( h = h_c = 1.0966 \),
and the flux rope current is \( J = J_c = 0.9924 \).
5.5.1 Prior to the Formation of the Current Sheet

The height at which the current sheet forms after the loss of equilibrium is determined by setting \( p = q = 0 \) in (5.21) and (5.7). These equations give

\[
J \ln \left( \frac{2h J}{r_{00}} \right) + \tan^{-1} \left( \frac{\lambda}{h} \right) = \ln \left( \frac{2\lambda_0}{r_{00}} \right) + \frac{\pi}{4}
\]

\[
J = \frac{h}{2\lambda},
\]

respectively, where \( r_{00} \) is the flux rope radius at \( \lambda = \lambda_0 = 1 \). Solving these two equations with \( r_{00} = 0.1 \) for \( h \) and \( J \) gives

\[
h^* = 1.8113
\]

\[
J^* = 0.9341,
\]

which are the values of \( h \) and \( J \) at the moment a neutral point appears at the base.

As the flux rope jumps from \( h = h_c \) to \( h = h^* \), a part of the stored magnetic energy is converted into the kinetic energy of the flux rope. In our model it is assumed that all of the energy released goes into kinetic energy. Therefore the velocity of the flux rope is just

\[
h = \sqrt{\frac{2(W_s - W(h))}{m}},
\]

(5.34)

where \( W_s \) is the total free energy per unit length stored in the system, i.e., \( W(h) \) at the critical point. This expression assumes that the mass is concentrated in the flux rope and neglects the fluid motions and waves from a distributed mass.
Forbes and Priest [1995] showed that

$$W(h) = \left(\frac{I_0}{c}\right)^2 \left[ J^2 \ln \left(\frac{2hJ}{r_{00}}\right) + \frac{1}{2} \frac{j^2}{J} \right].$$

Since the critical point corresponds to an unstable equilibrium, a perturbation of some kind is needed to make the system evolve. Here we use a small initial velocity for the flux rope of around 10 m/s, which is $\sim 10^{-5}$ times the Alfvén speed at $x = 0, y = 0$.

Substituting $\lambda_c, h_c$, and $J_c$ into (5.35) leads to $W_s = W(h_c) = 3.7766(I_0/c)^2$. Therefore, from (5.34) we obtained the velocity

$$\dot{h} = \dot{h}^* = 0.3211$$

of the flux rope at the moment that the current sheet forms. The parameters $\dot{h}^*, J^*$, and $\dot{h}^*$, together with the condition $p = q = 0$, provide the starting point for the integration of (5.33), which describe the evolution after the formation of a current sheet.

5.5.2 After the Formation of the Current Sheet

Taking a specific form for $\rho(y)$, such as $\rho(y) = \text{const}$ or that given by (5.32), we solve the simultaneous system of differential equations (5.33) with the initial condition (5.36) and with $M_A$ as a free parameter. In order to investigate the evolution of the system after the formation of a current sheet, we modify the equations in (5.33) into forms with time $t$ as the argument:

$$\frac{dp}{dt} = \frac{6}{5} p' \dot{h}$$
$$\frac{dq}{dt} = \frac{6}{5} q' \dot{h}$$
\[
\frac{d\dot{h}}{dt} = \frac{6}{5} \dot{h}' h \\
\frac{dh}{dt} = \frac{6}{5} \dot{h},
\]

(5.37)

where the time \( t \) is in minutes. We start counting the time at the moment when the catastrophic loss of mechanical equilibrium occurs. From (5.34), (5.35), and (5.36), it is easy to calculate that at \( t = 12.21 \) min, namely about 12 minutes after the system loses equilibrium, a neutral point appears at the boundary surface and a current sheet begins to develop. So, the initial conditions for the equations in (5.37) are

\[
\begin{align*}
t &= 12.2 \\
n &= 1.81 \\
\dot{h} &= 0.321 \\
p &= 0 \\
q &= 0. 
\end{align*}
\]

(5.38)

Plugging the equations in (5.33) into the equations in (5.37) and solving them under the initial conditions in (5.38), we obtain the variations of flux rope height \( h \), velocity \( \dot{h} \), current sheet parameters \( p \) and \( q \), etc., versus time \( t \).

If \( M_A = 0 \), we recover the results of Forbes and Priest [1995]. In the absence of reconnection and assuming no energy is dissipated, the flux rope oscillates (like a yo-yo) around an upper equilibrium at \( h \approx 8.9 \), and the highest location reached by flux rope is at \( h \approx 45.1 \) (i.e., the height where \( \dot{h} = 0 \) and the flux rope starts being pulled back).

If \( M_A > 0 \), escape is possible, but the flux rope may undergo several oscillations before escape occurs as shown in Figure 5-4, which shows the flux rope height and velocity as a
Figure 5-4: Variations of $h$, $\dot{h}$, $p$, and $q$ versus $t$ for reconnection in the isothermal atmosphere model with $M_A = 0.001$. The unrealistic oscillating behavior at early times indicates that this rate of reconnection is too slow to give a plausible result.

function of time for $M_A = 0.001$ for the isothermal atmosphere model with $\rho(y)$ given by (5.32). The reconnection is so slow that the flux rope oscillates in nearly the same manner as in the ideal-MHD case. However, on each oscillation the flux rope reaches greater height because of the ongoing erosion of the current sheet by reconnection. Finally, on the fourth bounce, the flux rope breaks free.

Since no CME has ever been observed to reverse its course and return to the Sun, oscillating behavior is not acceptable. To avoid oscillations, we have determined that $M_A$ must be greater than $\sim 0.005$. Figure 5-5 shows the trajectory for a case where $M_A$ exceeds this value. The flux rope moves continuously away from the Sun, but it does undergo a period of deceleration between $t = 20$ and $t = 100$. However, if $M_A$ exceeds 0.041, no deceleration ever occurs as shown by the example in Figure 5-6b for $M_A = 0.1$. 

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Figure 5-5: Variations of $h$, $\dot{h}$, $p$, and $q$ versus $t$ for reconnection in the isothermal atmosphere model with $M_A = 0.01$. Note that (a) and (b) have different timescales in order to show both short- and long-term behaviors. No oscillation occurs in this case, but the flux rope experiences a short period of deceleration from $t \approx 20$ min to $t \approx 90$ min.
Figure 5-6: Variations of $h$, $\dot{h}$, $p$, and $q$ versus $t$ for reconnections with $M_a = 0.1$: (a) the uniform atmosphere model and (b) the isothermal atmosphere model. Note that the scale of $p$ in panel (a) is expanded by 20 times.
Figure 5-7: The highest altitude the flux rope can reach (top curve) and the height at which deceleration occurs (bottom curve) as functions of the Alfvén Mach number $M_A$ for the model with an isothermal atmosphere.

By comparing Figures 5-5 and 5-6 with Figure 5-3a we can estimate how fast the flux rope and the tips of the current sheet are moving relatively to the local Alfvén speed. For the isothermal case shown in Figure 5-5 the flux rope's speed exceeds the local Alfvén speed within just a few minutes of the start of the eruption. (Judging from numerical experiments, the flux-rope's speed will be lower, by about a factor of 2, once the energy losses due to radiation, evaporation, and gravity are included.) For about the first 100 min the upper tip of the current sheet has nearly the same speed as the flux rope, so a quasi-steady treatment of the reconnection process within the sheet is not possible until $\sim 2$ hours after the start, at which time the speeds of $p$ and $q$ are both less than 5% of the local Alfvén speed.

Figure 5-7 summarizes the behaviors of the system as a function of $M_A$ for the isothermal atmosphere (i.e., an exponentially decreasing density with height). The top curve, extending
from $M_A = 0$ to $0.005$, shows the maximum height reached before the flux rope is pulled back. The bottom curve shows the height at which the force on the flux rope becomes negative and causes it to decelerate. The existence of the two critical Mach numbers, one for no oscillation and the other for no deceleration, depends on the fact that the Alfvén speed increases with heights at large altitude. If the Alfvén speed decreases with height, as in the uniform density model, then these critical Mach numbers no longer exist (Figure 5-8), and the flux rope is always pulled back to the Sun. This is because as the flux rope and current sheet become higher and longer, the local Alfvén speed at the current sheet tends to zero (Figure 5-3b), and thus, also the absolute rate of reconnection. However, even in this case, the maximum height reached by the flux rope for values of $M_A \geq 0.1$, is so large that the flux rope can be said to have escaped, since in a more realistic, nonplanar model, the flux rope would be entrapped by the solar wind at such a height.

5.6 Energy Release as a Function of Time

Without reconnection, only a small percentage ($< 8\%$) of the stored magnetic energy is converted into kinetic energy when flux rope loses equilibrium. Including reconnection greatly enhances the amount of energy that is released, but it also changes the time scale of the release process. In this section, we investigate how the reconnection affects the amount and rate of energy release.

Before the neutral point appears, there is no current sheet, and the energy release has nothing to do with reconnection. The output power is

$$ P = \frac{dW}{dt} = Fh, \quad (5.39) $$
where $F$ and $\hat{h}$ are given by (5.25) with $p = q = 0$ and by (5.34), respectively. Equation (5.39) gives the variations of $P$ as a function of $h$ before current sheet forms. To plot $P$ against $t$ instead of $h$, the time $t$ is obtained by integrating (5.34) with $W(h)$ determined by (5.35). This gives $t$ as a function of $h$ and makes it possible to plot $P$ as a function of $t$ by using $h$ as a parametric parameter.

As the current sheet starts developing, we have to integrate the differential equations in (5.37) from the time when the neutral point just appears at the boundary surface and then plug the results into (5.25). Finally, combining the equations for $P$ as a function of $h$ and that for $h$ as a function of $t$, we obtain the $P$ versus $t$ plot as shown in Figure 5-9. This power versus time curve exhibits three distinct phases. At first the rate of energy release is very low, but after $\sim 8$ min the rate escalates rapidly to a second phase with a very high

Figure 5-8: The same as Figure 5-7 but for the model with an atmosphere of uniform density.
Figure 5-9: Output power $P$ versus time $t$ for an isothermal atmosphere and for $M_{A0} = 0.1$. The inset shows the same result, but it has an expanded time scale to show better the initial evolution.

Power output that is sustained for only about a minute or so. A decline then ensues that leads to a moderately low output that lasts many hours. Overall, the behavior is similar to that for the light curves of long-duration events, but since we do not calculate the amount of magnetic energy that goes into radiation, the phases occurring in Figure 5-9 do not directly correspond to those of an actual light curve. Furthermore, since an X-type neutral line appears after the onset of the eruption, radiative features, such as chromospheric ribbons and X-ray loops, appear after the start of the eruption itself. In order to distinguish between the three phases of energy release curve and those of a light curve we will use the terms “precursor” to refer to the initial phase, “explosive” to refer to the second phase, and “late” to refer to the final phase.

Initially, the flux rope velocity is proportional to the perturbation velocity $\delta h$ used to
perturb the flux rope from its equilibrium position. We take $\delta \dot{h} \approx 10 \text{ m/s}$, corresponding to $10^{-5}$ of the ambient Alfvén speed. This value leads to a precursor phase of $\sim 8 \text{ min}$ followed by an explosive phase that begins a few minutes before the X-point appears. Once reconnection starts, the energy conversion increases, and the output power reaches its maximum at $\sim 15 \text{ min}$. Although reconnection is now important, the time scale of the energy release is still primarily the Alfvén time scale. Only after a large current sheet has formed does the reconnection time scale start to show in the energy release. If the reconnection were so fast that no current sheet formed, all the energy would be released in about an hour. However, with $M_A = 0.1$ as shown in Figure 5-9, the energy release takes more than 10 hours.

The model by Martens and Kuin [1989] also exhibits similar energy release behavior during the initial development. As in our model, they use a loss of equilibrium to trigger the eruption, but, unlike our model, they assume that the current sheet is much shorter than the global scale length of the field. Thus their model does not provide an adequate description of the field if the current sheet becomes long. For the case shown in Figure 5-9 the short current sheet approximation used by Martens and Kuin [1989] is valid for only $\sim 5 \text{ min}$ after the appearance of the current sheet because of the rapid rate at which the current sheet grows relative to the rate of reconnection. Thus, in this case, a more exact treatment of the current sheet is required in order to follow the long term evolution.

The difference between the rates of rise and decline in the energy release rate reflects the timescales of different physical processes. From our discussion above we know that the initial evolution is mainly driven by an ideal-MHD process, namely, the catastrophic loss of equilibrium that is governed by the Alfvén time scale. However, this process leads to the development of a current sheet that halts the further evolution unless reconnection occurs. So, the time scale of the subsequent evolution is determined by the rate of reconnection.
Since the reconnection time scale is long compared to the Alfvén time scale, we observe a long and slow decline. Thus we conclude that the difference between the rise time and decline time actually reflects the fundamental difference between the Alfvén time scale and the reconnection time scale.

5.7 Current Sheet Evolution

The local Alfvén speed is evaluated at the midpoint of the current sheet in our formulation. At the moment when the neutral point appears, we have a zero length current sheet at which the magnetic field, and thus the local Alfvén speed, is zero as shown in Figure 5-3b. This implies a zero reconnection rate, and a current sheet immediately forms following the appearance of the neutral point. It is only after the sheet has achieved a significant length that the local Alfvén speed shows the effect of the background field (see Figure 5-3a). Once the background field becomes important, it affects the length of the current sheet. In the isothermal model the Alfvén speed increases with height at large distance, so the absolute rate of reconnection \[ E_z = -(1/c)(\partial A^\theta/\partial t) \] increases with time. This increase explains why the current sheet grows shorter after about \( t = 5 \) hours in Figures 5-5 and 5-6b. By contrast, in the uniform density model the Alfvén speed decreases with height, so the absolute rate of reconnection always decreases with time. Consequently, the current sheet never grows shorter (Figure 5-6a).

The effect of the different reconnection rates is more noticeable in the position, \( p \), of the lower tip of the current sheet than in the position of the upper tip, \( q \). This is not so surprising since in the absence of reconnection, \( p \) remains equal to zero, while \( q \) does not. As the inset in Figure 5-5a shows, \( p \) initially rises at a rapid rate but then starts to slow with time. This pattern of motion is characteristic of post-flare loops, which are associated
Figure 5-10: A comparison between the trajectory of the giant X-ray arches created by a CME and the “post”-flare loops created by an eruptive flare (from Forbes and Lin [2000]).

with the reconnection process that occurs in the corona during flares.

Švestka et al. [1997] have noted that although the giant X-ray arches often generated after CMEs are similar in appearance to post-flare loops, they typically have a different pattern of motion, namely, at high altitudes, their upward velocity remains nearly constant or may even increase with time (Figure 5-10). Because of this different pattern of motion, there has been some doubt as to whether the giant arches could be explained by the same process as occurs in post-flare loops [Simnett and Forbes, 1991; Švestka et al., 1995]. Our model here suggests that they can and that the different motion they exhibit is because they lie at an altitude where the Alfvén speed is increasing with height, rather than decreasing as it does at low altitude. Consequently, the lower tip of the current sheet rises at an increasing speed because of the increase in the absolute rate of reconnection. Thus features like post-flare loops and giant arches that are associated with the lower tip of the current sheet show different patterns of motion because of the different altitudes at which they occur relative to the height at which the Alfvén speed starts to increase.
Another important aspect of the current sheet’s evolution is the behavior of the electric field in it. This field not only reflects the absolute rate of reconnection, but it may also be important for the acceleration of energetic particles. Figure 5-11 shows the electric field as a function of time for the isothermal model with $M_A = 0.1$. Since the current sheet does not appear until after the eruption has already started, the current sheet’s electric field is not preceded by a precursor phase. The field rises rapidly to its peak value within an Alfvén time scale and then decreases to a much lower rate that is sustained for a considerable time. The combination of a high peak field followed by a sustained low level field is suggestive of the production of the energetic particles inferred from X-rays and $\gamma$-rays for large, two-ribbon flares associated with CMEs. These eruptions produce a high output of energetic particles during their impulsive phase followed by a low-level output that is sustained for

Figure 5-11: The electric field at the current sheet as a function of time for the isothermal model with $M_A = 0.1$ (same case as in Figure 5-9).
many hours during the gradual phase [Kanbach et al., 1993].

5.8 Summary

In this chapter, we have constructed a flux rope model of a solar eruption that incorporates a feedback loop between the reconnection process behind the flux rope and the magnetic force which propels it outward. The feedback occurs because the propulsion force and the absolute reconnection rate \( E_z = M_A V_A B_y / c \) depend on each other. The model treats \( M_A \), the relative reconnection rate, as a free parameter and does not generally assume any particular model for the reconnection process itself. For illustrative purposes, we have assumed that \( M_A \) is a constant, less than unity, as has been proposed by Matthaeus and Lamkin [1986] for turbulent reconnection, but the model can easily be modified to incorporate values of \( M_A \), which are function of space and time. We assume that the magnetic field produced by currents flowing in the reconnection region are, at least at large distances from the region, well approximated by the field of an infinitely thin, vertical current sheet.

The two principal differences between our model and the previous one by Martens and Kuin [1989] are as follows: first, we have obtained an exact solution for the magnetic field rather than an approximate one. The exact solution allows us to follow the evolution even when the current sheet is large compared to the scale length of the photospheric field. Second, we have expressed the reconnection rate in the current sheet in terms of the inflow Alfvén number \( M_A \), instead of a constant sheet resistance as did Martens and Kuin [1989]. By expressing the reconnection rate in terms of \( M_A \) and requiring it to be less than unity, we can make sure that the inflow never becomes unphysical by exceeding the local Alfvén speed. Also, we can estimate the minimum value of \( M_A \) needed to give a plausible eruption, and thus, in principle, constrain reconnection theories which predict specific values of \( M_A \).
For the purpose of calculating quantities as a function of time we have assumed that the flux rope can be treated as a projectile. However, it is possible to calculate quantities, such as the positions of the upper and lower tips of the current sheet and the amount of magnetic energy released as a function of height without making this assumption. Treating the flux rope as a projectile overestimates the speed of the flux rope because all of magnetic energy that is released goes into the kinetic energy of the flux rope. In a more realistic model, much (perhaps as much as half) of the energy would be converted into heating and the wave energy associated with the generation of a fast-mode shock in front of the flux rope.

Our main conclusions can be summarized as follows:

1. Modest reconnection rates (as measured by $M_A$) are sufficient to allow the flux rope to escape into interplanetary space. For our most realistic model (the one with an isothermal atmosphere) a smooth escape occurs for all cases with $M_A > 0.005$. Thus the problem of opening the magnetic field by a purely ideal process that was first noted by Aly [1984] can be avoided if a small amount of reconnection is allowed to occur.

2. An extensive current sheet forms after an eruption even when $M_A$ is of order unity. The current sheet starts to shorten only during the late phase several hours after onset. For values of $M_A$ less than unity the lower tip of the current sheet rises so slowly, compared to the upper tip, that most of the magnetic field appears to become open to infinity before a substantial amount of reconnection has occurred. This behavior is consistent with that observed by the LASCO coronagraph on SOHO, namely, that the field lines appear to become fully opened before any X-point appears above 1.2 solar radii, the minimum altitude allowed by the SOHO occulting disk [Antiochos et al., 1999].

3. The model has the potential to explain the different patterns of motion seen in
post-flare loops and giant X-ray arches, if we assume that these structures occurs on the field lines mapping to, or just below, the lower tip of the current sheet. At the relatively low altitudes of post-flare loops, the model predicts that the rise rate of the loops should continually slow with time. However, at the relatively high altitudes of giant arches, the model predicts that the rise rate of the arches should increase with time. The transition between the two different patterns of motion occurs when the X-line reaches the altitude where the Alfvén speed of the ambient coronal field starts to increase with height due to the fall off in coronal density.

4. Finally, the model predicts that there should be a correlation between the output power of the eruption and the growth and decay of the current sheet. The quick drop of output power at the end of the explosive phase is related to the development of a long current sheet, while the long life time of the late phase is related to the extended time it takes for reconnection to dissipate the current sheet.
Chapter 6

Eruptions Triggered by Newly Emerging Flux

Using a simple model for eruptions, we investigate how an existing magnetic configuration containing a flux rope evolves in response to new emerging flux. Such an investigation is motivated by the suggestion of several researchers (e.g. *Feynman and Martin* [1995]; *Wang and Sheeley* [1999]; and *Canfield et al.* [1994]) that new emerging flux regions in the photosphere could destabilize a pre-existing flux rope or arcade. Our results show that the emergence of new flux can cause a loss of ideal MHD equilibrium under certain circumstances, but the circumstances which lead to eruption are much richer and more complicated than one might expect given the simplicity of the model that we discuss in this chapter. The model results suggest that the actual circumstances leading to an eruption are sensitive, not only to the polarity of the emerging region as previous researchers have proposed, but also to several other parameters, such as the strength, distance, and area of the emerging region. Based primarily on simple cartoon models of the connectivity of the field lines in the vicinity of a flux rope, various researchers have argued that the emergence of new flux with an orientation which allows reconnection with the pre-existing flux (a process sometimes referred to as *tether cutting*) will generally lead to destabilization of the coronal or prominence magnetic field. Although our results can replicate such behavior for certain restricted classes of boundary conditions, we find that in general there is no simple,
universal relation between the orientation of the emerging flux and the likelihood of an eruption.

6.1 Introduction

In the 1970's, new emerging flux was considered as possibly the main trigger of solar flares and perhaps also coronal mass ejections (CMEs) [Rust et al. 1980]. Some ground-based observations appeared to show a strong correlation between solar flares and the emergence of new magnetic flux [Rust 1972 and 1975]. Rust [1972] and his co-workers [Rust and Roy 1974; Rust et al. 1975; Rust and Bridge 1975] found that flares sometimes occur close to rapidly evolving magnetic features which emerge with their polarity reverse to that of the surrounding region [Rust 1973]. Their results also agree with the observations in Hα made by Vorpahl [1973]. Motivated by these observations, Heyvaerts et al. [1977], proposed a plausible model of solar flares, which has subsequently received a great deal of attention. In their model, a flare originates due to the sudden onset of magnetic reconnection between a newly emerging flux system and an old pre-existing flux system. The sudden onset is caused by the appearance of a plasma microinstability within the current sheet which greatly enhances the effective resistivity, but the exact nature of the microinstability is not prescribed. One may find more information about these earlier observations and models in the reviews by Švestka [1976 and 1981], Van Hoven et al. [1980], Sturrock [1980], and Zirin [1988].

Later in time, Martin et al. [1982 and 1984], showed that the apparent correlation between emerging flux and flares was not sufficiently strong to support the model of Heyvaerts et al. [1977]. Their studies seemed to suggest that something more than the onset of reconnection in a simple current sheet must be involved, thus the general interest in
the emerging flux as a flare-trigger waned. However, interest in emerging flux revived a few years ago, when *Feynman and Martin* [1995] presented new observational evidence that eruptions of quiescent prominences and associated CMEs sometimes occur as a consequence of the interaction between newly emerging active regions and the pre-existing large-scale magnetic field containing the prominence.

They observed, on the basis of more than 30 events, that prominence eruptions often occur shortly after the appearance of a new magnetic bipole in the vicinity of the prominence’s magnetic channel. They also found that the orientation of the bipole relative to the magnetic channel was not important if the bipole emerged within the channel, close to the polarity inversion line. However, the orientation was important if the bipole emerged outside the channel but still in the proximity of the prominence. For these latter cases they found an orientation which permits an X-line to form in the corona between the emerging bipole and the prominence (i.e. an orientation defined as “favorable” for reconnection) would usually lead to an eruption, while an orientation which did not permit an X-line to form (i.e. the orientation defined as “unfavorable” for reconnection) would usually not lead to an eruption. Not all of the events studied fit this pattern, as there were a few exceptions which showed that sometimes eruptions occurred without the presence of a nearby newly emerging flux, and that sometimes no eruption occurred despite the presence of newly emerging flux. Also, there were a few cases where eruption occurred even though the orientation was “unfavorable” and a few cases which failed to erupt even though the orientation was “favorable”.

Later on, *Wang and Sheeley* [1999] reported three examples of filament eruptions near newly forming bipolar magnetic regions which are consistent with the findings of *Feynman and Martin* [1995]. Several relatively recent studies on the general properties of emerging
flux in the photosphere have been completed by Canfield et al. [1996], Kurakawa and Santo [2000], Srivastava et al. [2000], Tang et al. [2000], and Chen and Shibata [2000].

In this chapter, we investigate how an eruption can be initiated by the destabilization of a force-free flux rope due to the emergence of new flux. In the next section, we summarize the previous flux rope model that is the basis of our analysis, and then we describe the magnetic configuration on which the present work is based. Section 6.3 presents the analysis of the magnetic configurations equilibrium in response to a newly emerging flux region. In section 6.4, we compare our results with the published observations, and finally, we summarize the results of the analysis in section 6.5.

6.2 Description of the Model

Our model applies the ideal-MHD equations for quasi-static equilibria to a two-dimensional configuration containing a flux rope. Realistically modeling the effect of an emerging flux region on a flux rope really requires a three-dimensional treatment since emerging flux regions are never observed to have the long linear structure that we assume. Thus the best we can hope to achieve with a model in which the emerging region, as well as the flux rope, are treated as two-dimensional is a basic understanding of the underlying process leading to eruption. For example, we can use a two-dimensional model to determine whether so-called "tether cutting" as a result of magnetic reconnection between an emerging and pre-existing flux systems can lead to eruption as has been proposed by Canfield et al. [1994] and Canfield and Reardon [1998].

In the absence of significant gas pressure gradient and gravity, equation (3.6) and relation (3.7) can be used to construct an evolutionary sequence of force-free equilibria in response to quasi-statically slow changes in the photosphere. That is, changes which occur on time-
scales much greater than the time for magnetoacoustic waves to cross the configuration.

6.2.1 Synopsis of Previous Model

The new model in this chapter is an extension of the previous one by Forbes and Isenberg [1991] for a flux rope magnetically suspended above a photosphere with the boundary condition (3.13). A current carrying flux rope of radius $r_0$ is used to model the magnetic cavity in which the prominence sits. Unlike the previous model by Forbes and Isenberg [1991], we will assume here that the photospheric boundary condition evolves at a rate which is much slower than the rate of reconnection in the corona so that no significant current sheet forms in the corona prior to eruption. Therefore, the sources of the coronal magnetic field are just the flux rope and the photospheric sources embodied by $A(x, 0)$. Equation (3.6) is replaced by (4.2), and $j_z$ just equals to $j_f$ in (4.4), namely

$$j_z(x, y) = I(h)\delta(x)\delta(y - h), \quad (6.1)$$

since the magnetic configuration that we are focusing on here does not contain the current sheet.

6.2.2 Magnetic Configurations with New Source

To include an emerging flux source, we modify the boundary condition (3.13) to

$$A(x, 0) = \frac{md}{x^2 + d^2} + \frac{sy_d}{(x - x_d)^2 + y_d^2}, \quad (6.2)$$

so that in addition to the two-dimensional line-dipole of strength $m$ located at $x = 0$, $y = -d$, we now have a second two-dimensional line-dipole of strength $s$ located at $x = x_d$, 

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\( y = y_d \), and function \( \phi(t) \) in boundary condition (3.13) which plays the role of describing the change of background field in system's evolution is now replaced by \( s, x_d \) and \( y_d \) in (6.2). The disappearance of the previous bilateral symmetry introduces a horizontal displacement \( x_h \) of the flux rope during the evolution in addition to the vertical displacement \( y_h \). This means that the line current (6.1) which represents the external field outside the flux rope becomes

\[
j_z(x, y) = I(x_h, y_h)\delta(x - x_h)\delta(y - y_h),
\]

where \((x_h, y_h)\) is the new position of the flux rope.

As in the previous model, the depth of the pre-existing dipole, \( d \), is chosen to be the characteristic length scale to which all lengths are normalized. Three other parameters, \( M, S, \) and \( I_0 \), are used to normalize \( m, s, \) and \( I \) in equations (6.2) and (6.3), such that

\[
MI_0 = mc/(4d), \quad SI_0 = sc/(4d), \quad \text{and} \quad I = JI_0,
\]

where \( I_0 \) has the dimension of current intensity, and \( M, S, \) and \( J \) are dimensionless parameters. Thus, equations (6.2) and (6.3) become

\[
A(x, 0) = \frac{4I_0}{c} \left[ \frac{M}{x^2 + 1} + \frac{Sy_d}{(x - x_d)^2 + y_d^2} \right],
\]

and

\[
j_z(x, y) = \frac{JI_0}{d^2} \delta(x - x_h)\delta(y - y_h).
\]

respectively. Substituting (6.4), (6.5), and (4.7) into (4.6) gives

\[
A(x, y) = \frac{2I_0}{c} \Re \left[ J \ln \left( \frac{z - z_h^*}{z - z_h} \right) + \frac{2iM}{z + i} + \frac{2iS}{z - x_d + iy_d} \right],
\]

where \( i = \sqrt{-1}, z = x + iy, z_h = x_h + iy_h, z_h^* = x_h - iy_h \), and \( \Re \) means taking the real part of the expression in square brackets. Figure 6-1 plots the contours of \( A(x, y) \) for a typical
configuration given by (6.6) with $M = 1$ for the pre-existing dipole located at $(0, -1)$, and $S = 1.5$ for the new emerging dipole located at $(3.25, -1.5)$. The equilibrium position of the flux rope is then located at $(1.8, 2)$.

6.2.3 Configurations in Equilibrium

In response to the change of the background field, the magnetic configuration in the model adjusts its equilibrium states successively. Prior to reaching a critical point, this adjustment, or evolution, is quasi-static and controlled by the photospheric motions with a time scale that is much longer than the magnetic reconnection time scale in the corona. Thus, the configuration may contain an X-point, but it does not contain a current sheet until the
magnetic configuration loses equilibrium.

Equilibrium requires that the total force acting on the flux rope vanishes. As long as the flux rope radius, \( r_0 \), is small compared to all other length scales, the external field \( \mathbf{B}_e \) (the total field \( \mathbf{B} \) minus the contribution from the flux rope) is effectively uniform within the flux rope. The condition for flux rope equilibria then dissociates into two separate conditions: one for the local equilibrium, which leads to (4.17) relating \( r_0 \) to \( J \), and one for the global equilibrium. The global equilibrium is obtained by setting the external force per unit length on the flux rope to zero. This force is given by

\[
\mathbf{F} = \frac{I_0}{c} J \hat{z} \times \mathbf{B}_e, \tag{6.7}
\]

where \( \hat{z} \) is the unit vector in \( z \) direction, and \( \mathbf{B}_e \) has the components

\[
\begin{align*}
B_{ex}(x, y) &= \frac{2I_0}{cd} \left\{ \frac{2M[x^2 - (y + 1)^2]}{[x^2 + (y + 1)^2]^2} + \frac{2S[(x - x_8)^2 - (y + y_8)^2]}{[(x - x_8)^2 + (y + y_8)^2]^2} \\
&\quad + \frac{J(y + y_8)}{(x - x_h)^2 + (y + y_h)^2} \right\}, \\
B_{ey}(x, y) &= \frac{2I_0}{cd} \left\{ \frac{4Mx(y + 1)}{[x^2 + (y + 1)^2]^2} + \frac{4S(x - x_d)(y + y_d)}{[(x - x_d)^2 + (y + y_d)^2]^2} \\
&\quad - \frac{J(x - x_h)}{(x - x_h)^2 + (y + y_h)^2} \right\}, \tag{6.8}
\end{align*}
\]

according to (3.7) and (6.6).

The global equilibrium is realized when \( B_{ex}(x_h, y_h) = B_{ey}(x_h, y_h) = 0 \), so finally we obtain the two conditions:

\[
\begin{align*}
\frac{2M[x_h^2 - (y_h + 1)^2]}{[x_h^2 + (y_h + 1)^2]^2} + \frac{2S[(x_h - x_d)^2 - (y_h + y_d)^2]}{[(x_h - x_d)^2 + (y_h + y_d)^2]^2} + \frac{J}{2y_h} &= 0, \tag{6.9} \\
\frac{M x_h(y_h + 1)}{[x_h^2 + (y_h + 1)^2]^2} + \frac{S(x_h - x_d)(y_h + y_d)}{[(x_h - x_d)^2 + (y_h + y_d)^2]^2} &= 0. \tag{6.10}
\end{align*}
\]
6.3 Evolution of Equilibria, Energetics and Bifurcation Set

Our main purpose is to find out how the equilibrium position of the flux rope, \((x_h, y_h)\), evolves in response to the gradual alteration of the photospheric background field, so that we can determine where catastrophes occur. To investigate the system's evolution in response to changes in the boundary condition (6.4), we need one more condition, namely the frozen-flux condition at the surface of flux rope. Thus, in addition to equations (6.9) and (6.10), we take

\[
\frac{2I_0}{c} A_R = A(x_h, y_h - r_0) = \text{const.} \tag{6.11}
\]

According to (6.6),

\[
A_R = J \ln \left( \frac{2y_h}{r_0} \right) + \frac{2M(y_h + 1)}{x_h^2 + (y_h + 1)^2} + \frac{2S(y_h + y_d)}{(x_h - x_d)^2 + (y_h + y_d)^2}. \tag{6.12}
\]

For simplicity, we determine the constant in (6.11) by calculating \(A_R\) in (6.12) as \(S = x_h = 0\) and \(J = y_h = M = 1\). The result leads to \(A_R = \ln(2/r_{00}) + 1\), which, in fact, is the value of \(A(x_h, y_h - r_0)\) when the flux rope is located at the maximum current point of the symmetric system \((S = x_h = 0\) and \(y_h = h)\). So, the frozen-flux condition (6.11) becomes

\[
J \ln \left( \frac{2y_h}{r_{00}} \right) + \frac{2M(y_h + 1)}{x_h^2 + (y_h + 1)^2} + \frac{2S(y_h + y_d)}{(x_h - x_d)^2 + (y_h + y_d)^2} = \ln \left( \frac{2}{r_{00}} \right) + 1, \tag{6.13}
\]

where equation (4.17) has been used to deduce (6.13) from (6.11) and (6.12). Equations (6.9), (6.10) and (6.13) are sufficient to determine the evolution of the equilibrium configurations.

As we will shortly make evident, the above system of equations leads to an extraordi-
narily complex behavior for the equilibrium properties of the system. Indeed, the increased
degree of complexity seems rather surprising to us given that all we have done is to introduce
a second photospheric source which is the same as the original source except for its strength
and location. However, we now have a system with two dependent variables, \( x_h \) and \( y_h \), and
five parameters, \( r_{00} \), \( M \), \( S \), \( x_d \), and \( y_d \), whereas before there was only a single dependent
variable, \( y_h \), and two parameters, \( r_{00} \) and \( M \). The much greater degree of complexity results
from this increase in the number of variables and parameters. According to the theory of
catastrophes developed by Thom [1972], a system with this number of variables corresponds
to a type of catastrophe known as the \textit{second order umbilic} which is the most complicated
type of catastrophe that his theory describes [Poston and Stewart 1978].

The equilibrium of a system occurs at the extrema and inflection points of the total
external energy \( E \), namely where \( \nabla E = 0 \) holds. In our system the total external energy \( E \)
is calculated by

\[
-dE = \mathbf{F} \cdot d\mathbf{r},
\]

where \( \mathbf{F} \) is the external force acting on the flux rope, as determined by (6.7), and \( \mathbf{r} = (x, y, z) \)
is the position vector. Substituting equation (6.7) into the above expression for \( dE \)
and modifying the results by using the frozen-flux condition (6.13), yields

\[
E_h = \left( \frac{I_0}{c} \right)^2 J^2 \left[ \ln \left( \frac{2y_h J}{r_{00}} \right) + \frac{1}{2} \right], \tag{6.14}
\]

which is the total external energy per unit length of the system when the flux rope is located
at the equilibrium position \( (x_h, y_h) \).

The equilibrium positions described by equations (6.9), (6.10), and (6.13) constitute a
manifold, or surface, in the multi-dimensional parameter space of the independent variables (i.e. $r_{00}$, $M$, $S$, $x_d$, and $y_d$). Catastrophes occur at locations where both $\nabla E_h = 0$ and

$$\frac{\partial^2 E_h}{\partial x_h^2} \frac{\partial^2 E_h}{\partial y_h^2} - \left( \frac{\partial^2 E_h}{\partial x_h \partial y_h} \right)^2 = 0$$

(6.15)

hold. Such catastrophes correspond to transitions from one local minimum to another, and they indicate a dynamic change of the system. After a somewhat lengthy manipulation, equation (6.15) simplifies to

$$\frac{4x_h^2}{(y_h + 1)^2} \left[ \frac{3(y_h + 1)^2 - x_h^2}{x_h^2 + (y_h + 1)^2} - \frac{y_h + 1}{y_h + y_d} \frac{3(y_h + y_d)^2 - (x_h - x_d)^2}{(x_h - x_d)^2 + (y_h + y_d)^2} \right]$$

$$+ 4 \left[ \frac{(y_h + 1)^2 - 3x_h^2}{x_h^2 + (y_h + 1)^2} - \frac{x_h}{x_h - x_d} \frac{(y_h + y_d)^2 - 3(x_h - x_d)^2}{(x_h - x_d)^2 + (y_h + y_d)^2} \right]$$

$$- \frac{1}{y_h(y_h + 1)} \frac{2 \ln(2y_h J/r_{00}) + 3}{\ln(2y_h J/r_{00}) + 1} \times \left\{ \frac{(y_h + 1)^2 - x_h^2}{x_h^2 + (y_h + 1)^2} - \frac{x_h(y_h + 1)[(y_h + y_d)^2 - (x_h - x_d)^2]}{(x_h - x_d)(y_h + y_d)} \right\}$$

$$\times \left\{ \frac{(y_h + 1)^2 - 3x_h^2}{x_h^2 + (y_h + 1)^2} - \frac{x_h}{x_h - x_d} \frac{(y_h + y_d)^2 - 3(x_h - x_d)^2}{(x_h - x_d)^2 + (y_h + y_d)^2} \right\} = 0.$$  

(6.16)

Equations (6.9), (6.10), and (6.13), provide the equilibrium, and when combined with equation (6.16), they determine the bifurcation set of the phase transitions. Equations (6.9), (6.10), and (6.13) can be thought of as prescribing a surface within a six-dimensional parameter space consisting of $y_h$ (or $x_h$), $r_{00}$, $M$, $S$, $x_d$, and $y_d$ as the coordinates, and the related bifurcation set is equivalent to a five-dimensional surface with $r_{00}$, $M$, $S$, $x_d$, and $y_d$ as the coordinates. These hyper-surfaces can only be plotted by taking slices in two or three dimensions or by using techniques which project them onto two or three dimensional space. One standard procedure used throughout the present work is to set $r_{00} = 0.01$. The effect of varying $r_{00}$ is discussed by Forbes and Isenberg [1991], Isenberg et al. [1993], Forbes.
et al. [1994], and Forbes and Priest [1995]. Generally, they found that varying $r_{00}$ produces minor effects when $r_{00}$ is assumed to be much smaller than the scale length. Since our analysis in this paper is only valid for small $r_{00}$, we take a small and fixed value for $r_{00}$.

In the case of $S = 0$, we are back to the previous situation studied by Forbes and Isenberg [1991]. In their work, the electrical conductivity was assumed to be infinitely high, the plasma behaved as a perfectly frozen-in medium, and the reconnection was completely forbidden. Therefore, a current sheet attached to the boundary surface could develop following the appearance of an X-point at the surface before the loss of equilibrium. The evolution of this system's equilibrium states was described by a three-dimensional surface ($y_h$ as a function of $M$ and $r_{00}$), and the most complicated catastrophic behavior that the system could exhibit is of the cusp type (i.e. an S-shaped equilibrium curve with two turning points. Refer to Figures 3-6 and 3-7). On the other hand, if we lift the unrealistic assumption of infinitely high conductivity of coronal plasma, a current sheet does not appear before the loss of equilibrium, the catastrophic behavior reduces to the simplest type, namely the fold (i.e. an equilibrium curve with a single turning point. Refer to Figures 6-2b, 6-3b, 6-4a, and 6-6c below).

However, as we return to the present case, namely $S \neq 0$, the equilibrium state of the system now needs six parameters in order to specify it. This leads to the most complicated type of catastrophe – the second-order umbilic [Thom 1972]. The complexity of the second order umbilic is so great that it is extremely difficult to describe its topology in any way that can be easily graphed. Rather than attempt to do so here, we refer the readers to Poston and Stewart [1978] who provide some limited examples of the topological prospects of the both first order and second order umbilic.

Before proceeding further, we point out the relative importance of $M$ and $S$. Although
Figure 6-2: Comparisons of the importance of the two sources $M$ (original dipole) and $S$ (newly emerged dipole). Variations of the equilibrium height of the flux rope, $y_h$, as a function of (a) both the strength of the original field source, $M$ and the newly emerging flux source, $S$; (b) $M$ only with $S = 0$, and (c) $S$ only with $M = 0$. The other parameters are set to $r_{00} = 0.01$ and $x_d = y_d = 1$. 

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we call $M$ the old source and $S$ the new source, they are interchangeable. This is shown by Figure 6-2a in which $y_h$ is plotted as a function of both $M$ and $S$, and also in Figures 6-2b and 6-2c in which $y_h$ is plotted as a function of $M$ for $S = 0$ and then as a function of $S$ for $M = 0$, respectively. When $S = 0$, equation (6.10) leads to $x_h = 0$, and the configuration is similar to that first considered by Van Tend and Kuperus [1978] (see also Kaastra [1985], Martens and Kuin [1989], and Van Ballegooijen and Martens [1989]). When $M = 0$, then $x_h = x_d$, and the configuration is identical to that of the $S = 0$ case except that it is now symmetric around $x = x_d$ instead of $x = 0$. Therefore, fixing the source $M$ and investigating the system's evolution in response to the variation of the source $S$ does not lose generality. Both plots in Figure 6-2b correspond to the simplest type of catastrophe, namely, the fold mentioned above. For the fold, the lower branch of the equilibrium curve corresponds to a stable equilibrium, while the upper one corresponds to an unstable one.

6.3.1 Modeling Emerging Flux by Strengthening a Source at Fixed Location

When the emergence of new flux is modeled by increasing the strength, $S$, its position coordinates $(x_d, -y_d)$ are treated as fixed parameters during the evolution of the photospheric field. A negative $y_d$ does not make any sense since the emerging flux cannot move out of the photosphere, so $y_d$ is always greater than zero. Which side of the flux rope the new flux appears on only affects on which side of the $y$-axis the equilibrium position is located, and neither the equilibrium height nor the evolutionary behavior of the system is affected because of the symmetry of the system. So, it is sufficient for us to consider only those cases for which $x_d$ is positive. With these restrictions, the equilibrium described by (6.9), (6.10), and (6.13) is now a function of a three-dimensional parameter space.
Figure 6-3: Variations of the equilibrium heights, $y_h$, as functions of both $S$ and $x_d$ (a). Variations of both coordinates, $x_h$ and $y_h$, of the equilibrium position versus $S$ with $x_d = 1$ (b), with $x_d = 3.25$ (c), with $x_d = 3.04$ (d), and with $x_d = 5$ (e). For the other parameters: $r_{00} = 0.01$, $y_d = 1.5$ and $M = 1$. NEF stands for New Emerging Flux.
Initially, we set \( y_d = 1.5 \), but other values will be considered later. This yields the equilibrium surfaces \( y_h(S, x_d) \) and \( x_h(S, x_d) \). Figure 6-3a plots the flux rope height \( y_h \) as a function of the lateral distance, \( x_d \), and the strength \( S \) of the emerging flux, and shows features typical of a catastrophe manifold, but it is much more complicated than the cusp-type manifold which appeared in the previous single source models [Forbes and Priest 1995]. In order to reveal the catastrophe behavior implied by Figure 6-3a, we slice this surface with a set of planes of \( x_d = \text{const} \), and thereby obtain the functional behavior of \( y_h \) and \( x_h \) as a function of \( S \) for different \( x_d \). The results are shown in Figures 6-3b through 6-3e for four different values of \( x_d \). Figure 6-3b corresponds to \( x_d = 1 \), so that the sources are in relatively close proximity to one another. In this case the magnetic fields of the sources blend together so that the two sources behave almost as one. Therefore, the system evolves in nearly the same way as for the simple fold catastrophe (see Figure 6-2). The only obvious difference is that the horizontal displacement of the equilibrium position, \( x_h \), is not zero, except for the \( S = 0 \) which corresponds to the old case of a single source.

When the separation, \( x_d \), becomes relatively large, as shown in Figure 6-3c with \( x_d = 3.25 \), the effect of the two separate sources become apparent. Now there are two independent equilibrium curves with distinctly different shapes. Before discussing the catastrophe behavior implied by these curves, we first show the transition case in Figure 6-3d which occurs when \( x_d = x_d^* = 3.04 \). At this critical value the two separate curves just touch one another at a single point.

In Figure 6-3c, the curves at the right are open, just as before, except that now the source \( S \) is the dominant one, rather than the source \( M \). The flux rope is located at \( x_h = x_d \) in the limit that \( S \) is very large. However, as \( S \) becomes weaker the effect of the source \( M \) creates the wiggles apparent in the range \( 1 < S < 2 \). Since \( S \) is always bigger than 1 along
the curve at right, it does not correspond to the case of newly emerging flux which requires $S$ to start at 0 and then slowly increase.

The emerging flux scenario is manifested by the curve at the left of Figure 6-3c which includes the point with $S = 0$. This curve is dominated by the source $M$, and the effect of $S$ is to perturb the system around the old equilibrium position that occurs when $S = 0$. This is also why $|x_A|$ of the equilibrium position of the flux rope is small.

The closed, or cyclic, equilibrium curve in Figure 6-3c has two critical points, and thus the quasi-statical evolution may end up with a loss of equilibrium no matter which way the system evolves along the equilibrium curve. In other words, if we start with an equilibrium corresponding to the configuration $S = 0$, the emergence of new flux with either positive or negative $S$ may cause a loss of equilibrium. For the case shown in Figure 6-3c the point $S = 0$ is located very roughly near the mid-point between the two critical points, so the probability of triggering an eruption is nearly the same for an emerging flux with negative polarity as it is for one with positive polarity. However, as $x_d$ (the lateral distance of the emerging source $S$) becomes larger, the situation changes as shown in Figure 6-3e. There are still two separate equilibrium curves, but the closed curve is now greatly elongated in the direction of negative $S$. This change makes the eruption much easier for a newly emerging flux with positive $S$ than for one with negative $S$.

The result that the importance of the sign of $S$ depends on $x_d$ is the same type of behavior that was observed by *Feynman and Martin* [1995] in their analysis of the effect of newly emerging flux on prominences, but, as we shall discuss in section 6.4, the dependence we find is more complex than the one they reported. For example, if we change the dipole depth from $y_d = 1.5$ (which we call the "shallow" source) to $y_d = 4.7$ (which we call the "deep" source) the behavior changes to that shown in Figure 6-4, which includes three
Figure 6-4: Equilibrium curves similar to those in Figures 6-3b through 6-3e, but with $y_d = 4.7$ and $x_d = 1$ (a), $x_d = 6$ (b) and $x_d = 8$ (c), respectively. The solid dot, A, indicate the critical point, and dots B and C show the possible positions to which the flux rope may approach after the loss of equilibrium. NEF stands for New Emerging Flux.
Figure 6-5: The magnetic configurations in (a) and (b) correspond to the two points A and B in Figure 6-4c, respectively. The "+" indicates the center of the flux rope.

different cases of $y_d = 4.7$, and $x_d = 1, 6, \text{ and } 8$, respectively.

Comparing Figure 6-4a ($x_d = 1$) with Figures 6-2a and 6-3b, we find that the system evolves in the same way as before for small $x_d$, but for a larger value of $x_d = 6$, the closed curve is now stretched in the opposite direction. Thus, the major part of the curve is now located in the region with $S > 0$, and only a tiny part of it is left in the region with $S < 0$. This behavior is the reverse of that reported by Feynman and Martin [1995] for distant sources.

As $x_d$ increases yet further to a value of 8, the equilibrium curves evolve to those shown in Figure 6-4c. At the critical point $A$, the flux rope can make a nearly horizontal jump to the point $B$, as shown in Figure 6-5. Alternatively, it may jump to infinity via point $C$. In the absence of a full energy analysis, it is not possible to determine the evolutionary path of the flux rope in the $x_h-y_h$ plane. It is quite possible, for example, the point $C$ is an unstable equilibrium point and the flux rope would never move toward it.
6.3.2 Modeling Emerging Flux by Moving a Source of Fixed Strength Upward

An alternate way to emerge magnetic flux is to move a source upward from below the photosphere by decreasing the source depth, $y_d$. During this process, the strength of the source may also change. But for simplicity, we only consider the case where the source strength, $S$, is fixed.

We start as before by drawing an equilibrium surface on a three dimensional space, but now with the parameter co-ordinates $y_d$, $S$, and $y_h$ instead of $x_d$, $S$, and $y_h$. Using equations (6.9), (6.10), and (6.13) we obtain the result shown in Figure 6-6a for $x_d = 3$. This surface is again a complicated one, but it looks distinctly different from the previous one.

To observe the evolutionary behavior, we slice this surface with planes of constant $S$ to obtain a set of equilibrium curves. The results are given in Figures 6-6b and 6-6c for $S = 1$ and 0.625, respectively. We also examined the case of $S < 0$ (Figure 6-6d), but no suitable behavior was found for $y_d$ within a reasonable range. Equilibria curves for $S > 1.2$ do not show any bifurcation.

The equilibrium curves in Figure 6-6b do not show any bifurcations which are suitable for the emerging flux scenario. There are nose points near $y_d = 1$, but as the source emerges, $y_h$ goes from a large value to a small one, so the flux rope would jump downwards instead of upwards were the system to move along this curve. Equations (6.9), (6.10), and (6.13) indicate that $y_d \to \infty$ is equivalent to $S \to 0$, so the right ends of these curves merge into the corresponding branches of the curves in Figure 6-2b. Therefore, the lower curves in Figure 6-6b are almost certain to be stable equilibria, and the upper ones are unstable equilibria.

For the next value of smaller $S$, we obtained the two equilibrium curves shown in Figure 6-6c. For this case a plausible eruption can occur. As the emerging flux approaches the
Figure 6-6: Equilibrium height, $y_h$, versus both $y_d$ and $S$ for $x_d = 3.0$ (a). Variations of $x_h$ and $y_h$ as functions of $y_d$ for $S = 1$ (b), $S = 0.625$ (c), and $S = -0.625$ (d).
Figure 6-7: Magnetic field configurations for the equilibria described by Figure 6-6 for $S = 0.625$, $y_d = 2$, $x_h = 0.12$ and $y_h = 0.84$. The "+" indicates the center of the flux rope.
surface, $y_d$ goes from a large value to a small one, and a flux rope located on the lower equilibrium curve at the right of the figure will move towards the nose point at 1.5. Once it reaches this point it jumps upward to infinity. Figure 6-7 shows the configuration of the magnetic field as it approaches the nose point.

The phase transition from the behavior shown in Figure 6-6b to that shown in Figure 6-6c occurs at $S = 0.741$. It is interesting that the likelihood of a loss of equilibrium actually decreases with the strength of the newly emerging flux rather than increasing as one might expect. This is because whether a bifurcation exists in the equilibrium curves is not simply a matter of the strength of the sources, but rather the balance between magnetic tension and pressure acting on the flux rope.

Finally, we consider the effect of increasing $x_d$ by changing it from 3 to 6. The system’s evolutionary behavior for $S = 1$ and $S = 3$ is shown in Figure 6-8. Both cases manifest the catastrophic feature. The case of $S = 1$ in Figure 6-6a is straightforward and it indicates the escape of the flux rope to infinity following the loss of equilibrium. However, Figure 6-6b for the case of $S = 3$ shows that the system has two equilibria within a small region near the critical point at around $y_d = 3.8$, as happened for the case described in Figure 6-4c. So, as the system evolves quasi-statically in response to the emergence of new flux (i.e. as $y_d$ decreases from $\infty$ to 3.8), the flux rope moves along the lower branch of the curves located at the right of each panel. The loss of equilibrium at the critical point causes the flux rope to jump to a new equilibrium position which is at a higher altitude and also closer to the new flux as shown in Figure 6-8c. In this case we have a loss of equilibrium, but we do not have escape, so this case does not corresponds to a CME. However, this case does illustrate the point that the same process which produces CMEs, can easily produce non-escaping flare-like events. Note also that in this case two current sheets may be created.
Figure 6-8: Equilibrium curves similar to those in Figure 6-6 but for $x_d = 6$: (a) $S = 1$ and (b) $S = 3$. The magnetic field configurations corresponding to the points $A$ and $B$ in (b) are drawn in (c) for $y_d = 3.65$. In the left panel the coordinates of the flux rope are $(0.2, 1.1)$ corresponding to point $A$, and in the right panel the coordinates are $(3.8, 4.75)$ corresponding to point $B$. The "+" in the panel at right indicates the center of the flux rope.
since there are two X-lines present before the loss of equilibrium.

We also checked the evolution when $S$ is larger than 3 as well as when $S$ is negative. We found that for $S > 4.07$, there are no nose point on the equilibrium curves, and that for $S < 0$, there are no equilibria for $0 \leq y_d \leq 70$.

### 6.3.3 Effect of the Horizontal Motion of The Source

Although the horizontal motion of the magnetic sources does not correspond to the emergence of new flux, it is of some interest because rapid horizontal motion is a well-known precursor for some types of flares (see Tanaka and Nakagawa [1973] and Martres et al. [1986]). It has even been suggested that rapid sunspot motion may be regarded as a flare precursor [Dezső et al., 1984; Kovács and Dezso, 1986; and Dezso and Kovacs, 1998].

Before considering the general case of an arbitrary separation distance, $x_d$, between the sources $M$ and $S$, we first look at the two extreme cases, $x_d = 0$ and $x_d \to \infty$. The case with $x_d = 0$ is already familiar to us since it is the same as the single source model with the strength $S + M$, in place of just $M$. The case with $x_d \to \infty$, has two equilibria, one with the flux rope located at $(x_h = 0, y_h)$, and the other located at $(x_h = x_d, y_h)$. These two equilibria are located exactly above their respective sources $M$ and $S$ which are completely decoupled from one another by their infinite separation.

Now let us consider the in between situations, where we take $y_d = 0$, and the source is just located at the boundary surface. We will consider two cases, namely $S = 1$ (Figure 6-9a) and $S = -1$ (Figure 6-9b). Figure 6-9a specifies two types of equilibrium curves: the lower one shows bifurcation, while the upper one does not. Note also that the bifurcation occurs only when the two sources are relatively far apart. Since the upper curve completely covers the lower one, a loss of equilibrium corresponds to a primarily horizontal jump from
Figure 6-9: Variations of both $x_h$ and $y_h$, of the equilibrium position of flux rope versus $x_d$ with $S = 1$ (a), and $S = -1$ (b), respectively. The other parameters are: $r_{oo} = 0.01$, $y_d = 0$ and $M = 1$.

The old equilibrium above the source $M$ to a new equilibrium above the source $S$. This is similar to the case shown in Figures 6-4c. Thus this case is also not a good candidate for a description of a prominence eruption or a CME. However, if we set $S = -1$, instead of 1, we get the case shown in Figure 6-9b, which does seem reasonable. Equations (6.9), (6.10) and (6.13) indicate that for this case, in the limit $x_d \to \infty$, the equilibrium tends to $(x_h = 0, y_h)$. Therefore, the lower branch of the $y_h$ curve and the upper branch of $x_h$ curve correspond to the stable branch of the equilibrium curve. Consequently, the loss of equilibrium occurs here by means of a nearly vertical jump to infinity which is the kind of behavior needed for an eruption.
6.4 Comparisons with Observations

Statistical analyses of observational data show that around 70% of sudden prominence disappearances are closely correlated with the emergence of new flux [Bruzek 1952; Feynman and Martin 1995]. The results given by Feynman and Martin [1995] further indicate that the likelihood of an eruption becomes higher when the newly emerging flux has an orientation favorable for reconnection with respect to the pre-existing filament. The words "favorable for reconnection" are used to describe the case when there is an X-point located between the old source and the new source, and it corresponds to positive $S$ in our notation. To explore more details of this relationship, Wang and Sheeley [1999] investigated three filament eruptions observed with EIT on SOHO. Although this is not a statistically significant set, their results are consistent with those of Feynman and Martin [1995], and they provide more data on the evolution of the photospheric field leading up to the eruption.

In our work presented in this chapter, we have obtained many theoretical outputs from our model, some of them are consistent with the observations and others are not. Although the distance, $x_d$, between the new flux and the old one is in principle observable, neither the strength of the emerging dipole, $S$, nor its depth, $y_d$, is. To make the comparisons of our results with observations easier, we follow a suggestion by J. Klimchuk (private communication), and transform $S$ and $y_d$ into two observable parameters, namely the net change, $\delta A$, in the total flux and the area (per unit length) $W$ of the new flux on the surface.

From equation (6.4), the surface fluxes corresponding to $M$ and $S$ are $4I_0 M/c$ and $4I_0 S/(c y_d)$, respectively. So, the relative change of the total flux due to $S$ can be defined, in a simple way, as

$$\delta A = \frac{S}{M y_d}.$$ (6.17)
which can be either positive or negative depending on the sign of $S$. We use the total half-width, $W$, of the new flux distribution function to describe the area per unit length (i.e. the width of the emerging flux region in the 2D model) of the new flux on the surface. From equation (6.4), we have simply

$$ W = 2y_d, \quad (6.18) $$

which describes the concentration of the magnetic flux produced by $S$ on the boundary surface. When the source is deep, the flux spreads out over a large region of the surface, and $W$ is large. When the source is shallow, the flux concentrates within a small region of the surface, and $W$ is small.

By measuring $\delta A$ and $W$, one can discriminate between various scenarios for flux emergence. If only a change in $\delta A$ is measured, then the new flux is implemented by a strengthening source at a fixed depth. However, if both $W$ and $\delta A$ change, but their product, $W\delta A$, does not, then the new flux is implemented by the rising of a source with fixed strength. If all three quantities (i.e. $\delta A$, $W$, and $W\delta A$) change, then both the depth and strength are altered. Of course, all this assumes that $M$ is fixed, and that the sources are in the form of two dimensional dipoles.

Table 6.1 summarizes our results in terms of the change, $\delta A$, of the total flux and the width (area per unit length) $W$. The flux change, $\delta A$, is given as a percentage of the pre-existing flux, and the width, $W$, is normalized to the scale length, $d$. Altogether, the results for 11 different cases are tabulated, the first six correspond to variations of $S$ for three different values of the horizontal separation, $x_d$, and two different values of the source depth, $y_d$. For these six cases (1 through 6) only solutions with nose points on curves intersecting

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Table 6.1: Summary of eruptive behaviors found in the emerging flux model

<table>
<thead>
<tr>
<th>Case</th>
<th>Figure</th>
<th>$S$</th>
<th>$x_d$</th>
<th>$y_d$</th>
<th>$W$</th>
<th>$\delta A$ (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>6-3b</td>
<td>variable</td>
<td>1.0</td>
<td>1.5</td>
<td>3.0</td>
<td>-0.76</td>
</tr>
<tr>
<td>2</td>
<td>6-3c</td>
<td>variable</td>
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<td>1.5</td>
<td>3.0</td>
<td>-46</td>
</tr>
<tr>
<td>3</td>
<td>6-3e</td>
<td>variable</td>
<td>5.0</td>
<td>1.5</td>
<td>3.0</td>
<td>-675</td>
</tr>
<tr>
<td>4</td>
<td>6-4a</td>
<td>variable</td>
<td>1.0</td>
<td>4.7</td>
<td>9.4</td>
<td>-0.73</td>
</tr>
<tr>
<td>5</td>
<td>6-4b</td>
<td>variable</td>
<td>6.0</td>
<td>4.7</td>
<td>9.4</td>
<td>-5.2</td>
</tr>
<tr>
<td>6</td>
<td>6-4c</td>
<td>variable</td>
<td>8.0</td>
<td>4.7</td>
<td>9.4</td>
<td>-23</td>
</tr>
<tr>
<td>7</td>
<td>6-6b</td>
<td>1.0</td>
<td>3.0</td>
<td>variable</td>
<td>2.0</td>
<td>100</td>
</tr>
<tr>
<td>8</td>
<td>6-6c</td>
<td>0.625</td>
<td>3.0</td>
<td>variable</td>
<td>3.0</td>
<td>-41.6</td>
</tr>
<tr>
<td>9</td>
<td>6-6d</td>
<td>-0.625</td>
<td>3.0</td>
<td>variable</td>
<td>83.5</td>
<td>-1.5</td>
</tr>
<tr>
<td>10</td>
<td>6-8a</td>
<td>1.0</td>
<td>6.0</td>
<td>variable</td>
<td>6.5</td>
<td>30.7</td>
</tr>
<tr>
<td>11</td>
<td>6-8b</td>
<td>3.0</td>
<td>6.0</td>
<td>variable</td>
<td>7.2</td>
<td>83.3</td>
</tr>
</tbody>
</table>

$^a$ Flux rope jumps downward in this case.

$^b$ This case corresponds to an implausibly diffuse source.

The $S = 0$ line are listed, because only these curves fit the emerging flux scenario. The last five cases in Table 6.1 correspond to variations of the depth, $y_d$, for four different values of $S$ and two different values of $x_d$. For these last five cases (7 through 11) only solutions with nose points on curves which extend outward to $y_d = \infty$ are listed, because these are the curves which fit the emerging flux scenario when $S$ is held fixed and only $y_d$ is varied.

For the first six cases, holding $y_d$ fixed is equivalent to keeping the area per unit length (i.e. width) of the emerging flux region constant because $W$ is just twice $y_d$. Therefore, for these cases variations in $S$ translate directly into changes in the total flux $\delta A$. The value of $\delta A$ listed for each entry in the last column of Table 6.1 corresponds to the amount of flux which has emerged at the nose point of the equilibrium curve. The entries which have two values of $\delta A$ correspond to the type of closed equilibrium curve shown in Figure 6-3c, while entries which have only one value of $\delta A$ correspond to the type of open equilibrium curve shown in Figure 6-3b. When there are two values of $\delta A$, one value is always positive and the other is always negative. The positive value corresponds to an emerging flux region whose polarity is said to be favorable for reconnection, while the negative value corresponds to a region whose polarity is unfavorable for reconnection. Thus, the sign of $\delta A$ indicates...
whether reconnection will occur between the old and new flux system, while the magnitude of $\delta A$ indicates how much flux of a given sign is required to trigger an eruption.

For the last five cases in Table 6.1, both $W$ and $\delta A$ vary as $y_d$ is varied with $S$ held fixed, but the variation of $W$ is generally less than that of $\delta A$. Also for these cases, only one value of $\delta A$ is possible because the equilibrium curve cannot be a closed loop as was possible for those cases with $y_d$ held fixed and $S$ varying. The reason for this is as follows. When $S$ is held fixed and $y_d$ is varied, no emerged flux corresponds to having the source $S$ infinitely far below the surface, since this location corresponds to $y_d = \infty$.

Table 6.1 confirms the point we made earlier that, despite the apparent simplicity of our model, no simple pattern emerges from it with regards to the polarity of the emerging region and the likelihood of an eruption. By themselves, cases 3, 8 and 10 would seem to agree with the observed behavior reported by *Feynman and Martin* [1995] that the emergence of flux with an orientation favorable for reconnection increases the likelihood of a prominence eruption. But when we look at some of the other cases in the table, we see that their behavior is not consistent with this observation. In cases 1 and 4, for example, only the emergence of bipolar region with a polarity unfavorable for reconnection leads to eruption. *Feynman and Martin* [1995] also reported that the distance of the emerging flux from the filament channel was important. If the new flux appeared within the channel, then either polarity (favorable or unfavorable) would trigger an eruption, and it was only when the emergence occurred outside the channel the polarity was important. However, in our model there is again no single relation between the distance $x_d$ of the emerging flux and the importance of the emerging bipoles' polarity, although there is some tendency for such a behavior to occur, for example, in cases 2 and 3.

We also noticed that cases 7 through 11 imply that reasonable eruptions can only be
triggered by emerging flux with a favorable polarity, but never by an unfavorable one. The fact that this behavior is quite different than the cases 1 through 6, corresponding to a changing $S$, suggests that observations of eruptive behaviors might be used to determine how the newly emerging flux is produced. In other words, the behavior produced by increasing the strength of a source at a fixed location (cases 1 through 6) is different than the behavior produced by moving a source with the fixed strength upward (cases 7 through 11). For the former, the relation of the likelihood of an eruption to the polarity of the emerged flux is not simple, but for the latter, the relation is relatively straightforward since only the emerged flux with a favorable polarity triggers an eruption. This suggests that the behavior observed by Feynman and Martin [1995] and Wang and Sheeley [1999] might be because of the way new flux emerges, rather than because of the way reconnection affects the balance of forces acting on the flux rope.

6.5 Summary

In this chapter we have presented a flux rope model of solar eruptions that incorporates the process of flux emergence. We started with the quasi-steady MHD model of Forbes and Isenberg [1991], which is based on concepts first introduced by Van Tend and Kuperus [1978] and then developed by Martens and Kuin [1989], Van Ballegooijen and Martens [1989]. In this model there is a flux rope which is suspended in the corona by a balance between magnetic compression and tension forces. By slowly evolving the photospheric magnetic field, it is possible to create an imbalance which causes the flux rope to be ejected upwards. Here we have used a newly emerging flux system to evolve the boundary conditions. Altogether, there are four free parameters which describe the boundary conditions and one free parameter (the radius) which describes the flux rope. Because of this large number of free
parameters, the equilibria constitute multi-dimensional surfaces with complex geometry, and the number of different scenarios one can construct to obtain an eruption is very large. Thus, despite the apparent simplicity of our two-dimensional model, it actually leads to a quite complex behavior. So much so, it is difficult to extract any simple rule for predicting what types of emerging flux will produce an eruption. This is even more likely to be the case for a more realistic three-dimensional system, which has potentially many more degrees of freedom.

Thus, the simple picture that new flux with an orientation favorable for reconnection will trigger an eruption by the “tether cutting” of the field lines overlying the flux rope is insufficient to describe the behavior we observed. In hindsight this does not seem particularly surprising within the context of a flux-rope model based on a loss of ideal-MHD equilibrium. The occurrence of catastrophic behavior in such a model is not caused by simply decreasing the magnetic tension holding the flux rope in place or by increasing the magnetic compression pushing it upward. Normally, as one of these forces is decreased or increased, the other automatically compensates so that equilibrium is maintained and no catastrophe occurs. It is only at special locations (i.e. the nose points of the equilibrium curves) where the forces cannot balance and catastrophe can occur. Reconnection between the new and old flux systems may, or may not, drive the system to one of these nose points. Although we can replicate the results of *Feynman and Martin* [1995] to some extent, we obtain additional possibilities for eruption that they did not report. It may be that these additional possibilities can account for those few cases they observed which did not fit the general pattern. However, without a more detailed examination of such cases and a more realistic three-dimensional model, it is not possible to determine whether this is so.

The detailed behavior of our system can be summarized as follows:
If the new emerging flux is modeled simply by strengthening the source, $S$, the occurrence of catastrophic behavior depends on the source’s location, $x_d$ and $y_d$, as follows:

For small $x_d$ ($\simeq 1$), changing the source depth, $y_d$, has little effect and the evolution corresponds to a simple fold catastrophe, but eruptions occur only if the orientation of the new flux is unfavorable for reconnection (i.e. $S < 0$). This corresponds to cases 1 and 4 in Table 1.

As $x_d$ increases beyond a critical value, the catastrophic behavior changes. Either orientation of new flux (i.e. favorable with $S > 0$ or unfavorable with $S < 0$) may lead to an eruption, but the source depth, $y_d$, now starts to have an effect. For a shallow source (small $y_d$ as in cases 2 and 3), a new flux with a favorable orientation is more likely to lead to an eruption than one with an unfavorable orientation. This behavior is consistent with the observations of Feynman and Martin [1995] and Wang and Sheeley [1999]. For a source located deeper in the photosphere (large $y_d$ as in cases 5 and 6), the situation is reversed, so that a new flux with an unfavorable orientation can trigger an eruption more easily than one with a favorable orientation. This is just opposite to the observations.

If the new emerging flux is modeled by a fixed strength magnetic source rising from below the photosphere (decreasing $y_d$), then the results are as follows:

(i) For an orientation unfavorable for reconnection ($S < 0$), no equilibrium exists for large $|S|$ and large $x_d$. Equilibria do exist for small $|S|$ and small $x_d$, but catastrophes occur only for an implausibly diffuse source (case 9 in Table 6.1).

(ii) For an orientation favorable for reconnection (positive $S$), equilibria exist, and plausible catastrophes occur for reasonable values of $S$ (cases 7, 8, 10, and 11).

For the related case of purely horizontal motion of a source of fixed strength, catastrophes exist for $S$ of either polarity, but the evolutionary behavior of the two cases are not the
same. The catastrophes which occur for positive $S$ (i.e. favorable for reconnection), cause the flux rope to jump mostly in the horizontal direction from a location above the old source to one above the new source. These catastrophes are not reasonable representations of a typical solar eruption. By contrast, for negative $S$, primarily vertical jumps occur, which reasonably models the behavior expected for a solar eruption.

Finally, the magnetic system of our model may lose ideal-MHD equilibrium catastrophically in many situations. But for some of them, such as those described in Figures 6-4c and 6-9a, it is not clear if either catastrophe represents a major eruption. One way to determine whether a particular catastrophe would be suitable is to use our model results as a starting point for numerical simulations. However, the catastrophes shown in Figures 6-3b, 6-3c, and 6-6c, do not have another equilibrium curve located right above critical points, so, these cases are excellent candidates to produce a CME type eruption.
Chapter 7

Effect of Curvature on Flux-Rope Models of Eruptions

The large scale curvature of a flux rope can help propel it outwards from the Sun. This curvature force or "hoop" force, as it is sometimes called, is caused by the pinches of the poloidal field near the edge of the flux rope where the flux rope is bent. In other words, the flux rope resists any effect to bend it. In any three-dimensional model of the solar eruption, the flux rope must be curved since it must be of finite length and connected either to itself or to the surface.

Here we extend previous two-dimensional flux-rope models of coronal mass ejections to include the curvature force. To obtain analytical results we assume axial symmetry and model the flux rope as a torus which encircles the Sun, although in reality the flux rope is more likely to be attached to the photosphere. Initially, the flux rope is suspended in the corona by a balance between magnetic tension, compression, and curvature forces, but this balance is lost if the photospheric sources of the coronal field slowly decay with time. To present the photospheric field in the absence of the flux rope, we use a three-dimensional Sun-centered dipole. As we will show in this chapter, the maximum total magnetic energy which can be stored before equilibrium is lost is 1.53 times the energy of the potential field, and this value is less than the theoretical limiting value of 1.662 for the fully opened field. As a consequence, the loss of ideal-MHD equilibrium which occurs in the model
cannot completely open the magnetic field. However, the loss of equilibrium does lead to the formation of a current sheet, and if rapid magnetic reconnection occurs in this sheet, the flux rope can escape from the Sun.

7.1 Introduction

Previously, we concentrated on the two-dimensional system with Cartesian coordinates. That system evolved quasi-statically in response to changes of the photospheric background field, and the equilibrium curve contained one or more critical points where quasi-static evolution was no longer possible. It is at such a point that eruption is expected to occur. In order to investigate a more realistic configuration, we now turn to a system with axial symmetry. Unlike the previous system, the present system has a finite scale and thus a finite total energy. Therefore, we are able to determine whether the Aly-Sturrock limit applies to this system which contains non-simply connected magnetic arcades.

In this chapter, we modify the previous analyses by replacing a flat solar surface and an infinitely long cylindrical flux rope with a spherical solar surface of radius $R_0$ and a circular flux rope of major radius (or the height from the center of the Sun) $h$ and cross-sectional (minor) radius $r_0$ around the Sun, respectively. Although this model is still geometrically oversimplified compared to real eruptions, it contains two important improvements over the previous models. First, it includes the curvature force which is likely to be significant in driving the CMEs outward from the Sun, and second, the total energy stored is now finite instead of infinite as in the previous models. As before, $r_0$ is still assumed to be much smaller than the height, $h$, of the flux rope, that is $r_0 \ll h$.

Although our axisymmetrical model now includes curvature effects, it still has the feature that the ends of the flux-rope are not anchored to the photosphere. Therefore, the possibility
remains that if the ends were anchored, the catastrophic behavior we find in the present solution might be eliminated. However, there is no evidence at the present time that anchoring the ends of the flux rope prevents catastrophe behavior.

In the next section, we outline the model and in section 7.3, we solve the governing equations for a configuration without current sheets. In section 7.4, we obtain asymptotic solutions for systems with current sheets in the limit that the flux rope is far from the Sun. Finally, we summarize and discuss the physical meaning of our results in section 7.5.

7.2 Description of The Model

The model consists of two-dimensional equilibrium solutions in a spherical system with axial symmetry, and with \( r = R_0 \) being the photospheric boundary. The coronal region \( r > R_0 \) is assumed to be perfectly conducting, so that magnetic field lines are frozen to the plasma. A toroidal force-free magnetic flux-rope is suspended in the corona at the height, \( h \), by a balance between tension, compression, and curvature forces. The latter occurs when a straight, current-carrying flux rope is bent into a curve [Shafranov 1966]. In the present work, we do not address the question of how a flux-rope might be formed, but several mechanisms have been suggested in previous studies (e.g. Van Ballegooijen and Martens [1989]; Inhester et al. [1992]; and Van Ballegooijen [1999]).

A catastrophic jump may occur before or after a neutral point (or X-point) appears, depending on the structure of the photospheric field and on the radius of the flux rope. Here we assume that the reconnection time scale is long compared to the coronal Alfvén time scale, but short compared to the time scale of the photospheric field evolution. This means that current sheets continually dissipate as they try to form during the slow evolution phase prior to eruption. However, during the eruptive phase, the system evolves so rapidly that
current sheets can form because the evolution is too fast for them to dissipate.

The coronal magnetic field during the storage phase is determined by the normal field component at the photospheric boundary, and the magnetohydrostatic equations:

\[
\frac{1}{c} \mathbf{J} \times \mathbf{B} = \nabla p + \mathbf{F}_g, \quad (7.1)
\]

\[
\nabla \times \mathbf{B} = \frac{4\pi}{c} \mathbf{J}, \quad (7.2)
\]

\[
\nabla \cdot \mathbf{B} = 0, \quad (7.3)
\]

where \( \mathbf{B} \) is the magnetic field, \( \mathbf{J} \) is the current density, \( p \) is the plasma gas pressure, \( \mathbf{F}_g \) is gravitational force and \( c \) is the speed of light.

By invoking the force-free approximation, we assume that both the gravitational and pressure forces in (7.1) are negligible. Although these forces may be quantitatively important for some aspects of solar eruptions, they are not essential to the catastrophe behavior we consider here. This leads to

\[
\mathbf{J} \times \mathbf{B} = 0. \quad (7.4)
\]

Solving (7.2) and (7.4) with the appropriate boundary condition gives the whole magnetic configuration of the system. We start by choosing a spherical coordinate system and putting its origin at the center of the Sun with the \( z \)-axis in the north direction, so, (7.2) and (7.4) can be combined and simplified to one equation:

\[
\nabla^2 A_\phi - \frac{A_\phi}{r^2 \sin^2 \theta} + \frac{1}{2r \sin \theta} \frac{d(rB_\theta \sin \theta)^2}{d(rA_\phi \sin \theta)} = 0, \quad (7.5)
\]

where, \( B_\theta \) is the field in azimuthal direction, and \( A_\phi \) is the \( \phi \) component of the vector.
potential $\mathbf{A}$ defined by

$$(B_r, B_\theta, B_\phi) = \left[ -\frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta A_\phi), -\frac{1}{r} \frac{\partial}{\partial r} (r A_\phi), -\frac{1}{r} \frac{\partial}{\partial r} (r A_\theta) - \frac{1}{r} \frac{\partial A_r}{\partial \theta} \right] \quad (7.6)$$

in this geometry, and $A_r$, $A_\theta$ and $B_\phi$ all vanish outside the flux rope. Note the differences of equations (7.5) and (7.6) from their Cartesian counterparts (3.6) and (3.7).

### 7.3 Configuration Prior to Eruption

For our analysis, equation (7.5) can be re-written as a Poisson equation (see Appendix B):

$$\nabla^2 (A_\phi \cos \phi) = -\frac{4\pi}{c} J_\phi \cos \phi. \quad (7.7)$$

where $J_\phi$ includes the current density in the flux rope and also the current sheet, if the latter exists. We further assume that there are no magnetic field sources at infinity, that the flux-rope is located in the equatorial plane, that the flux rope is the only current source in the region $r > R_0$ prior to the eruption, and that the minor radius, $r_0$, of the flux rope is much smaller than its distance $h$ from the Sun. The boundary condition at the photosphere represents a field equivalent to that produced by a point dipole at the center of the Sun. We choose a point dipole to represent the boundary field not because we think it is the best one to use, but because it allows us to compare our analysis to previous ones which have used such a boundary condition (e.g., Mikić and Linker [1994]; Low and Smith [1993]; Roumeliotis [1992]). In fact, a previous analysis of the 2-D system with translational symmetry suggests that the dipole boundary condition is the least effective in producing a vigorous eruption (see Forbes et al. [1994]).
The field in the region outside the flux-rope can most easily be determined by representing the flux rope as a circular line current of the form

\[ J_\phi(r, \theta) = \frac{I}{h} \delta(\cos \theta) \delta(r - h), \]  

(7.8)

where \( I \) is the current strength carried by the flux-rope. Note that this is an approximation which is valid only outside the flux rope, since within the flux rope, \( J_\phi \) is finite and \( J_\theta \) is not zero. The vector potential at the photosphere, which is just the background dipole field, is

\[ A_\phi(r = R_0, \theta) = \frac{m_0 \sigma(t) \sin \theta}{R_0^2}, \]  

(7.9)

where \( m_0 \) is a constant and \( \sigma(t) \) is a function which slowly varies with time. Thus, the evolution of the photospheric field described by \( \sigma(t) \) is similar to that in a decaying active region, except that here our assumed symmetry makes the variation global.

Equation (7.7) can be solved by means of Green's method by combining it with (7.8) and (7.9). The solution is

\[ A_\phi(r, \theta) \cos \phi = \frac{1}{c} \int_V J_\phi(r', \theta') \cos \phi' G(r, r') dV' - \frac{1}{4\pi} \oint_S A_\phi(r_0, \theta') \cos \phi' \frac{\partial G(r, r')}{\partial r'} |_{r'=r_0} dS', \]  

(7.10)

where the volume integral is taken over the whole region with \( r > R_0 \) and the surface integral is taken over the surface of the Sun, namely, \( r' = R_0 \). The function \( G(r, r') \) is the Green's function for our system:

\[ G(r, r') = \frac{1}{|r - r'|} - \frac{R_0}{r' |r - R_0^2 r'/r'^2|}. \]
which in terms of spherical coordinates can be written as:

\[
G(r, r') = \frac{1}{(r^2 + r'^2 - 2rr'\cos\gamma)^{1/2}} - \frac{1}{(r^2r'^2/R_0^2 + R_0^2 - 2rr'\cos\gamma)^{1/2}},
\]

(7.11)

where \(\gamma\) is the angle between \(r\) and \(r'\) and \(\cos\gamma = \cos\theta \cos\theta' + \sin\theta \sin\theta' \cos(\phi - \phi')\), and \(G(r, r')\) satisfies

\[
G(r, r')|_{r=R_0} = G(r, r')|_{r'=R_0} = 0.
\]

Note the differences between equation (7.11) and its Cartesian counterpart (4.7).

Substituting (7.9), (7.8) and (7.11) into (7.10) gives:

\[
A_\phi(r, \theta) = \frac{4I ho}{c\sqrt{h^2 + r^2 + 2hr \sin \theta}} \left[ \frac{(2 - k^2)K(k) - 2\kappa(k)}{k^2} \right] - \frac{4I h R_0}{c\sqrt{h^2r^2 + R_0^4 + 2hrR_0^2 \sin \theta}} \left[ \frac{(2 - \tilde{k}^2)K(\tilde{k}) - 2\tilde{\kappa}(\tilde{k})}{\tilde{k}^2} \right] + \frac{m_0\sigma(t)}{r^2} \sin \theta,
\]

(7.12)

where \(K\) and \(E\) are complete elliptic integrals of the first and second kind, respectively, and

\[
k^2 = \frac{4hr \sin \theta}{h^2 + r^2 + 2hr \sin \theta}, \quad \tilde{k}^2 = \frac{4hrR_0^2 \sin \theta}{h^2r^2 + R_0^4 + 2hrR_0^2 \sin \theta}.
\]

In a spherical coordinate system with axial symmetry, magnetic field lines are described by

\[
\frac{dr}{B_r} = \frac{r d\theta}{B_\theta},
\]
Figure 7-1: Magnetic field configuration in a meridional plane ($r A_\phi \sin \theta$ contours) when the system is at the critical point for the case of $r_{00} = 0.01$. Tick marks are spaced every solar radius in the equatorial and polar directions.

which yields

$$r A_\phi \sin \theta = \text{const.},$$

(7.13)

holding (with a different constant) on each field line. Figure 7-1 shows the magnetic field contours of our model in a meridional plane.
7.3.1 Global Equilibria

According to (7.6), the result given in (7.12) yields a magnetic field on the \( \theta = \pi/2 \) (equatorial) plane in terms of the series:

\[
B_\theta(r, \theta = \pi/2) = \frac{4\pi I}{ch} \sum_{n=0}^{\infty} (n + 1) \left[ \frac{(2n + 1)!!}{(2n + 2)!!} \right]^2 \left[ f(r, n) - \frac{h^2}{R_0 r} \left( \frac{R_0^2}{r h} \right)^{2n+2} \right] + \frac{m_0 \sigma(t)}{r^3},
\]

(7.14)

where

\[
f(r, n) = \begin{cases} 
\frac{2n+2}{2n+1} \left( \frac{r}{h} \right)^{2n} & \text{if } r < h \\
\left( \frac{h}{r} \right)^{2n+3} & \text{if } r > h,
\end{cases}
\]

which corresponds to the field produced by the current in the flux rope alone. By subtracting \( f(r, n) \) and setting \( r = h \) in (7.14), we can deduce the external magnetic field, \( B_e \), acting on the flux-rope due to the sources outside of it. The result is

\[
B_e = -\frac{4\pi I}{cR_0} \sum_{n=0}^{\infty} (n + 1) \left[ \frac{(2n + 1)!!}{(2n + 2)!!} \right]^2 p^{2n+2} + \frac{m_0 \sigma(t)}{h^3} = \frac{4I}{cR_0} \left[ K(p) - \frac{E(p)}{1 - p^2} \right] + \frac{m_0 \sigma(t)}{h^3},
\]

(7.15)

where \( p = (R_0/h)^2 \). Note that the external magnetic field, \( B_e \), includes not only the contribution of the background dipole field (the third term at the right hand side of equation [7.15]) but also that of a surface current (the other terms at the right hand side of [7.15]) which is equivalent to an image of the flux rope beneath the surface.

In a two-dimensional system with translational symmetry, the body forces on the flux rope are all proportional to \( B_e \), so the equilibria are situated at the positions where \( B_e \)
vanishes (refer to the discussions in the previous chapters). However, this condition does not hold in the present case. The curvature of the flux-rope creates an additional outward force which must be balanced by the external field, $B_e$. This force was calculated by Shafranov [1966], and the magnitude is equal to the current, $I$, times the magnetic field $B_s$:

$$B_s = \frac{I}{c h} \left( \ln \frac{8 h}{r_0} - 1 \right), \quad (7.16)$$

where $r_0$ is the minor radius of the toroidal flux-rope and we have assumed that the field inside the flux-rope is prescribed by Lundquist's force-free solution (see Appendix C). For a force-free flux-rope, this force is always directed outward in the radial direction, and it is caused by the compression of the poloidal field along the inside edge of the torus.

Matching $B_e$ in (7.15) and $B_s$ in (7.16) gives the condition for global equilibrium, namely

$$4 J \left[ K(p) - \frac{E(p)}{1 - p^2} \right] + \frac{\sigma u}{h^3} = \frac{J}{h} \left( \ln \frac{8 h}{r_0} - 1 \right), \quad (7.17)$$

where now all lengths are normalized to $R_0$, the radius of the Sun, $J$ is the current normalized to $I_0 = m_0 c / (u R_0^2)$, and $p = 1/h^2$. The constant $u$ is used to normalize $\sigma$ to unity when $J$ reaches one, its maximum value.

Since the frozen-flux condition requires the magnetic field lines at the surface of the flux-rope to remain unchanged, we require, according to (7.13), that

$$(h - r_0) A_R = (h - r_0) A_\phi(h - r_0, \frac{\pi}{2}) = \text{const.}$$
Substituting (7.12) into this equation gives:

\[ h A_R = \frac{I_0}{c} h \left\{ 2 J \ln \frac{8 h}{r_0} - 4 J + 4 J h [E(p) - K(p)] + \frac{\sigma n}{h^2} \right\} = \text{const.} \quad (7.18) \]

to the lowest order of \( r_0/h \). To determine \( h \) solely as a function of \( \sigma \), we need to consider the local equilibrium inside the flux-rope.

### 7.3.2 Internal Equilibrium

The distribution of current density and field inside the flux-rope is arbitrary to some extent since we are free to choose any force-free configuration initially in the same way that we are free to choose a photospheric boundary condition.

For our model, we will use Lundquist's [1950] solution which has been extensively used in modeling coronal mass ejections (e.g., Kumar and Rust [1996]). For simplicity, the calculation of the internal equilibrium is performed in a cylindrical coordinate system where \( B_z \) is along the central axis, \( B_\theta \) is the azimuthal component, and \( B_\rho \) is the radial component relative to the axis. In this system, Lundquist’s solution is

\[
\begin{align*}
B_\rho &= 0 \\
B_\theta &= B_0 J_1(\lambda r) \\
B_z &= B_0 J_0(\lambda r),
\end{align*}
\]

(7.19)

The \( J_0 \) and \( J_1 \) are the zeroth and first order Bessel functions, and \( \lambda \) is a constant.

For a toroidal flux-rope the Lundquist solution is approximately correct as long as the minor radius, \( r_0 \), is much smaller than the major radius, \( h \) [Miller and Turner 1981]. Since the flux-rope in our model is located in the equatorial plane, the external field \( B_e \) does not
contribute to the toroidal component within the flux-rope. This means that the conservation of toroidal flux simply gives

\[ \Phi_t = \int_0^{2\pi} \int_0^{r_0} B_z \rho d\rho d\theta = \text{const.} \quad (7.20) \]

Since we also have \( \lambda B_z = 4\pi j_z / c \) (the force-free condition), equation (7.20) reduces to

\[ \Phi_t = \frac{4\pi}{\lambda c} \int_0^{2\pi} \int_0^{r_0} j_z \rho d\rho d\theta = \frac{4\pi I}{\lambda c} \]

with

\[ \frac{I}{\lambda} = \text{const.} \quad (7.21) \]

where \( I \) is the total current in the flux-rope.

At the surface of the flux-rope, \( B_z \) is equal to zero so \( J_0(\lambda r_0) = 0 \) there. Hence, \( \lambda r_0 \) is the first zero of \( J_0 \), namely \( \lambda r_0 = 2.405 \). Substituting this into (7.21), we find

\[ r_0 I = \text{const.} \quad (7.22) \]

which determines the relationship between the flux-rope radius \( r_0 \) and the current \( I \). Equation (7.22) has the same form as (4.17) although they were deduced under different conditions.

### 7.3.3 Evolution of the System

Equations (7.17) and (7.18) indicate that equilibria can only exist for flux-rope currents less than a particular value. This can be seen by using equation (7.17) to eliminate \( \sigma u \) in
equation (7.18), differentiating the result with respect to $h$, and setting $dJ/dh = 0$ with

$$r_0 = \frac{r_{00} I_0}{I} = \frac{r_{00}}{J},$$

(7.23)

where $I_0$ and $J$ are defined as before. At the current maximum, the flux-rope equilibrium height $h_0$ is related to $r_{00}$ by

$$3 \ln \frac{8h_0}{r_{00}} - 2 + 8h_0 \left[ \frac{2 - 4p_0^2}{(1 - p_0^2)^2} E(p_0) + \frac{3p_0^2 - 2}{1 - p_0^2} K(p_0) \right] = 0,$$

(7.24)

where $p_0 = 1/h_0^2$. We will refer to this state, $h = h_0$, as the maximum current point, and $r_{00}$ corresponds to the radius of the flux-rope at this point. When $r_{00}$ vanishes, $h_0$ goes to unity.

At the maximum current point, we set the constant on the right-hand side of (7.18) equal to $h_0 A_R^0 / c$ and use (7.17) to eliminate $\sigma u$ in (7.18), giving

$$h A_R = h_0 A_R^0 \frac{I_0}{c}$$

$$= h_0 \frac{I_0}{c} \left\{ 3 \ln \frac{8h_0}{r_{00}} - 5 + 4h_0 \left[ \frac{2 - p_0^2}{1 - p_0^2} E(p_0) - 2K(p_0) \right] \right\}.$$ 

Thus

$$A_R^0 = 3 \ln \frac{8h_0}{r_{00}} - 5 + 4h_0 \left[ \frac{2 - p_0^2}{1 - p_0^2} E(p_0) - 2K(p_0) \right],$$

(7.25)

with this explicit expression for the constant in (7.18), we have

$$J = \frac{A_R^0 h_0}{h} \left\{ 3 \ln \frac{8h}{r_0} - 5 + 4h \left[ \frac{2 - p^2}{1 - p^2} E(p) - 2K(p) \right] \right\}^{-1}.$$ 

(7.26)
Returning to (7.17) at the maximum current point, we obtain

\[ u = h_0^2 \left( \ln \frac{8h_0}{r_0} - 1 \right) + 4h_0^2 \left[ \frac{E(p_0)}{1 - p_0} - K(p_0) \right]. \]  \hspace{1cm} (7.27)

The current \( J \) is determined self-consistently from the frozen-flux condition (7.18) which depends on \( r_0 \), the radius of the flux rope at the maximum current point. As mentioned before, \( \sigma \) is unchanged during an eruption (or catastrophe), therefore the \( h-\sigma \) curve reveals where a catastrophic loss of equilibrium can occur. Substituting (7.26) into (7.17) to eliminate \( J \) leads to

\[ \sigma = \frac{hh_0A_H}{u} \frac{\ln \frac{8h}{r_0} - 1 + 4h \left[ \frac{E(p)}{1 - p^2} - K(p) \right]}{3 \ln \frac{8h}{r_0} - 5 + 4h \left[ \frac{2-p^2}{1-p^2} E(p) - 2K(p) \right]}, \]  \hspace{1cm} (7.28)

which describes the flux-rope equilibrium height \( h \) as a function of the photospheric boundary condition parameter \( \sigma \).

The dependence of \( h \) as a function of \( \sigma \) is illustrated in Figure 7-2 for \( r_0 = 10^{-2} \). The equilibrium curve in Figure 7-2 exhibits the switchback shape characteristic of a catastrophic system. To model an eruption, we suppose that the flux-rope starts somewhere on the portion of the curve which is close to the solar surface, for example, at \( \sigma = 1, h = 1.186 \). At this location the flux rope is stable to displacements in any direction [Forbes 1990b and 1991], but if the photospheric field slowly weakens with time (corresponding to decreasing \( \sigma \)), a stable equilibrium will eventually become impossible. As \( \sigma \) decreases, the flux rope moves leftward and upwards along the equilibrium curve until it reaches the lower nose point at \( \sigma = 0.9458, h = 1.340 \). Once this critical point is reached, the forces pushing the flux rope away from the Sun can no longer be balanced by the magnetic tension which holds the flux rope down. Thus, at the critical point (i.e. nose point), the flux rope loses
equilibrium and is ejected outwards, away from the Sun. As the flux rope moves outward, the horizontal field at the equator decreases until an X-type neutral line appears. In the ideal-MHD limit, the field around the neutral line immediately collapses to a current sheet that grows in length as the flux rope moves farther from the Sun. However, as we will show in the next section, the increasing magnetic tension associated with the growth of the sheet eventually stops the outward movement of the flux rope and leads to the formation of a new equilibrium configuration without opening the field.

Due to mathematical difficulty of solving Poisson’s equation (7.7) for a mixed boundary condition (i.e. the solar surface plus current sheet), we are not able to determine a general
analytical solution which includes a current sheet of arbitrary size. Thus, we can only be sure that an eruption occurs when the critical point \( h_c \) appears before the formation of a current sheet. The location of the critical point is determined by setting \( d\sigma/dh = 0 \) in equation (7.28), while the location where a current sheet forms is determined by setting \( B_\theta(r = R_0 = 1, \pi/2) = 0 \) in equation (7.14) in conjunction with the equilibrium condition (7.28). We label this location as \( \sigma_s(h_s) \), and it is given by

\[
\sigma_s = \frac{A_h^0 h_0}{u} \left\{ \ln(8Jh_2/r_\infty) + 2h_2[E(p_s) - K(p_s)] - 2 \right\}^{-1} \left\{ \frac{1}{h_s} \right\}, \tag{7.29}
\]

where \( p_s = 1/h_s^2 \). Figure 7-3 plots the variations of \( (h_c - h_s) \) versus \( r_\infty \), from which we see that for sufficiently small \( r_\infty \), a current sheet forms before catastrophe occurs. The radius at which this occurs is \( r_\infty \approx 2 \times 10^{-3} \).

Figure 7-1 shows field lines \((rA_\phi \sin \theta \text{ contours})\) for the equilibrium configuration at the critical point for the case of \( r_\infty = 0.01 \). Although \( h_c \) is less than \( h_{ne} \) in this case, the fractional difference \( (h_s - h_c)/h_c \) is only 1.1%. This is reflected in Figure 7-1 by the fact that the X-point lies just below the photosphere at \( \theta = \pi/2 \).

### 7.3.4 Energetics

One of the important aspects of the model is to evaluate the free magnetic energy stored in the system before catastrophe occurs. The stored magnetic energy of the system before catastrophe is equal to the work required to move the flux-rope from infinity to the critical point. Thus, the energy is given by

\[
W = -\int_P F(h, \Phi)dh + \frac{1}{c}\int_P I(h, \Phi)d\Phi, \tag{7.30}
\]
Figure 7-3: Comparison of two types of equilibrium heights, namely, the critical point height (the equilibrium height of the flux-rope when it is at the critical point — dashed curve), and the current sheet formation height (the equilibrium height of the flux-rope when an X-line appears at surface of the Sun — solid curve). The two heights are shown as functions of flux-rope radius $r_{oo}$, and are equal to each other at around $r_{oo} = 2.5 \times 10^{-3}$.

where the force $F(h, \Phi)$ is

$$F(h, \Phi) = 2\pi h \frac{I}{c} (B_s - B_e)$$

$$= 2\pi \left(\frac{I_0}{c}\right)^2 J \left\{4Jh \left[K(p) - \frac{E(p)}{1 - p^2}\right] + \frac{\sigma u}{h^2} - J \left(ln \frac{8Jh}{r_{oo}} - 1 \right)\right\}, \quad (7.31)$$

the flux rope current $I = I_0 J$ is given by (7.26), the magnetic flux $\Phi$ between the flux rope surface and the photosphere is:

$$\Phi = hA_R - R_0A_0 \quad (7.32)$$
where \( A_\theta = A_\phi (R_0, \pi/2) \), \( P \) is a path of integration in the \( h-\sigma \) plane from infinity (where \( J = 0 \)) to the critical point. The work done in evolving from one state to another is independent of the path \( P \) taken between the two states. For simplicity, we choose \( P \) such that \( d\Phi = 0 \), and then the second integral in (7.30) vanishes.

We calculate the energy of the configuration at height \( h \) by computing the work required to move the flux rope from infinity to \( h \). This process need not be ideal, so we can assume that reconnection at any X-line is so rapid that no current sheet forms (see Isenberg et al. [1993]). The energy \( W \) of any state without a current sheet is then given by (7.30) and (7.31) with \( \sigma \) held constant. Therefore, defining \( W^* \) as the energy at the critical point, we find that the free energy stored in the system prior to eruption is

\[
W^* = 2\pi \left( \frac{I_0}{c} \right)^2 J_c^2 h_c \left\{ 2h_c[E(p_c) - K(p_c)] + \left( \ln \frac{8J_c h_c}{r_0^\sigma} - \frac{3}{2} \right) \right\}
- \pi \left( \frac{I_0}{c} \right)^2 \int_{h_c}^{\infty} J^2 dh,
\]

(7.33)

where \( J_c \) is \( J \) at the critical point, and \( p_c = 1/h_c^2 \).

To facilitate comparison with previous studies, we also calculate the total magnetic energy of the system which is defined as \( W_{\text{total}} = W_{\text{potential}} + W \), where \( W_{\text{potential}} \) is just the potential energy associated with the photospheric field sources. At the critical point:

\[
W_{\text{potential}}^* = \frac{1}{3} \left( \frac{I_0}{c} \right)^2 (\sigma_c u)^2,
\]

(7.34)

where \( \sigma_c \) is \( \sigma \) at this point. Figure 7-4 plots \( W_{\text{total}}^*/W_{\text{potential}}^* \) against \( r_{00} \), the flux-rope radius at the point of maximum current on the equilibrium curve. The figure shows that although \( W_{\text{total}}^*/W_{\text{potential}}^* \) increases with \( r_{00} \), it remains less than 1.662, the ratio for a fully opened Sun-centered dipole field [Aly 1984; Sturrock 1991 and Low and Smith 1993].

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Figure 7-4: The ratio of total magnetic energy to potential magnetic energy at the critical point as a function of the flux-rope radius $r_{oo}$. It is not known if there are any catastrophes for $r_{oo} < 2.5 \times 10^{-3}$ (dashed portion of curve), because the stability of the equilibria in this region has not been determined. Note that the stored energy is always less than that of the fully opened field (horizontal dash-dot line).

The solid curve in Figure 7-4 corresponds to configurations of the system at critical points without an X-line, and when these configurations erupt, they generate current sheets which are attached to the solar surface. By contrast, the dashed curve corresponds to the configurations containing an X-line prior to eruption. Such configurations occur if $r_{oo}$ is less than $2.5 \times 10^{-3}$, and if we assume that the coronal reconnection time scale is much smaller than the time scale for the evolution of the photospheric field. Thus, unlike the situation during the eruption of the flux-rope, the appearance of an X-line during the slow evolution along the equilibrium curve does not lead to the formation of a large-scale current sheet. (Analyses of reconnection rate in CMEs/flares by Poletto and Kopp [1986], Lin et al. [1995], and Lin and Forbes [2000] and X-ray bright points by Parnell et al. [1994] indicate...
that the coronal reconnection time scale is longer than the Alfvén time scale but shorter than the convective time scale for the evolution of the photospheric field.

When a configuration containing an X-line erupts, a current sheet immediately appears. Consequently, in the absence of any practical method for solving our equations when a current sheet of finite length is present, we cannot analyze the ideal-MHD stability of the configurations lying on the dashed curve in Figure 7-4. Although we know equilibria exist at these locations, we do not know whether an upward vertical displacement will create an inward restoring force or an outward driving force.

Further, we notice that in the present model, because of the curvature of the flux rope, systems with flux ropes of small radius have less energy than those with flux ropes of large radius — behavior which is opposite to the previous models with translational symmetry. This difference is due to the fact that the curvature force becomes infinitely large as the radius, $r_0$, tends to zero. Therefore, to have equilibrium as $r_0$ tends to zero, the current $J$ in the flux-rope must also tend to zero. However, as the current goes to zero, so does the free magnetic energy, and thus, the net effect of the curvature force at small radii is to reduce the amount of free energy stored in the system.

### 7.4 Asymptotic Behavior of The System

If a current sheet of an arbitrary size exists, the boundary value problem for Poisson’s equation (7.7) becomes much more difficult. At the surface of the sphere, the boundary condition is still

$$A_\phi(R_0, \theta, \phi) = \frac{m_0 \sigma(t) \sin \theta}{R_0^2}$$  \hspace{1cm} (7.35)
but on the current sheet

\[ r A_\phi(r, \pi/2, \phi) = R_0 A_0 \quad R_0 \leq r \leq q, \]  

(7.36)

where \( A_0 = m_0 \sigma / R_0^3 = I_0 \sigma u/c. \)

Due to the additional complexity introduced by the disk-geometry of the current sheet, we can no longer obtain an analytical solution, although a numerical solution is possible, at least, in principle [Tur and Priest 1985]. However, we can investigate the asymptotic behavior of the system when the current sheet becomes very large. This case occurs when the flux-rope is far from the Sun, and the length, \( q, \) of current sheet, is much larger than \( R_0. \) In this limit the Sun can be regarded as a point, and the boundary condition (7.35) can be ignored since at large distance the dipole contribution becomes negligible compared to that of the current sheet. The boundary-value problem now consists of (7.7) with \( J_\phi \) given in (7.8) and the boundary condition on the disk of current prescribed by (7.36). The solution is

\[
A_\phi = -\frac{\pi I_\hbar}{c} \sum_{n=0}^{\infty} \frac{(-1)^n(2n-1)!!}{2^n(n+1)!} \frac{r_<^{2n+1}}{r_>^{2n+2}} P_{2n+1}^1(\cos \theta) \\
+ \sum_{n=0}^{\infty} A_{2n+1} P_{2n+1}^1(\xi) Q_{2n+1}^1(i \cdot \zeta), 
\]

(7.37)

where \( P \) and \( Q \) are Legendre functions of the first and second kinds, \( r_< = \min(h, r), \) and \( r_> = \max(h, r). \) The variables \( \zeta \) and \( \xi \) are the oblate spheroidal coordinates which are related to \((x, y, z)\) and \((r, \theta, \phi)\) as:

\[
x = r \sin \theta \cos \phi = q \sqrt{(1 + \zeta^2)(1 - \xi^2)} \cos \phi \\
y = r \sin \theta \sin \phi = q \sqrt{(1 + \zeta^2)(1 - \xi^2)} \sin \phi
\]
\[ z = r \cos \theta = q \zeta \xi, \quad (7.38) \]

with \(0 \leq \zeta < \infty\) and \(|\xi| \leq 1\). Here \(q\) is the radius of the disk (or current sheet) and

\[
A_{2n+1} = (-1)^n \frac{4n + 3}{(n+1)(2n+1)} \left[ - \frac{A_0 R_0}{\pi q} \frac{(2n)!!}{(2n+1)!!} \right]
+ \frac{2I}{c} (1 + \zeta_0^2)^{1/2} \frac{(2n-1)!!}{(2n+2)!!} Q_{2n+1}(i \cdot \zeta_0).
\quad (7.39)
\]

with \(\zeta_0 = \sqrt{(h/q)^2 - 1}\). Consequently, the magnetic field configuration is given by \(r \sin \theta A_\phi = \text{const.}\), as shown in Figure 7-5, which corresponds to the limit of the open state of the system.

By taking the curl of (7.37) and evaluating at \(\theta = \pi/2\), we obtain the field in the equatorial plane, namely,

\[
\begin{align*}
B_r &= 0 \\
B_\theta &= 0 \\
B_r &= f_0(r) + \frac{2}{q \zeta} \left\{ \frac{A_0 R_0}{\pi q} \left[ \frac{1}{1 + \zeta^2} \right] \right. \\
&\quad + \left. \frac{I}{c} \sqrt{1 + \zeta_0^2} \sum_{k=0}^{\infty} \frac{4k + 3}{(k+1)(2k+1)} \left[ \frac{(2k+1)!!}{(2k)!!} \right]^2 Q_{2k+1}(i \cdot \zeta_0)Q_{2k+1}(i \cdot \zeta) \right\},
\end{align*}
\quad (7.40)
\]

for \(r \geq q\), where

\[
f_0(r) = \frac{4I}{ch} \left\{ \begin{array}{ll}
-E \left( \frac{h}{r} \right)/ \left[ 1 - \left( \frac{h}{r} \right)^2 \right] & \text{if } r < h \\
\frac{h}{r} \left\{ E \left( \frac{h}{r} \right)/ \left[ 1 - \left( \frac{h}{r} \right)^2 \right] - K \left( \frac{h}{r} \right) \right\} & \text{if } r > h,
\end{array} \right.
\]

and \(\zeta = \sqrt{(r/q)^2 - 1}\). Note that the magnetic field goes to infinity at the tip of the current sheet, \(r = q\) or \(\zeta = 0\) in (7.40), unless the following condition is satisfied:

\[
\frac{A_0 R_0}{\pi q} = \frac{I}{c} \sqrt{1 + \zeta_0^2} \sum_{k=0}^{\infty} \frac{(-1)^k(4k + 3)}{(k+1)(2k+1)} \frac{(2k+1)!!}{(2k)!!} Q_{2k+1}(i \cdot \zeta_0).
\]

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Figure 7-5: Asymptotic magnetic field configuration in a meridional plane ($rA_\phi \sin \theta$ contours) when the current sheet length is much larger than a solar radius. The case shown is for $r_{00} = 0.01$, and field lines near the axis of the flux-rope (indicated by dots) are not drawn.

\[ B_\phi = \frac{I}{c} \frac{2s}{\sqrt{1 - s^2}}, \quad (7.41) \]

where $s = q/h$. This condition forces $B_\theta$ to vanish at the tip of current sheet and form a Y-point there as shown in Figure 7-5 in order to keep the whole current sheet in a force-free state.

Also note that the first term on the right-hand side of (7.37) or (7.40) is due to the flux-rope alone, and that subtracting this term simply gives the external field $B_e$:

\[ B_e = \frac{2I}{c h \zeta_0} \left\{ \frac{2s^2}{\sqrt{1 - s^2}} + (1 + \zeta_0^2) \sum_{k=0}^{\infty} \frac{4k + 3}{(k + 1)(2k + 1)} \left[ \frac{(2k + 1)!!}{(2k)!!} \right]^2 Q_{2k+1}^1(i \cdot \zeta_0)Q_{2k+1}(i \cdot \zeta_0) \right\} \]
\[
= \frac{I}{\varepsilon h} \left( \frac{2s^3}{1 - s^2} - 2s + \ln \frac{1 + s}{1 - s} \right),
\]

where (7.41) has been used. The balance between \( B_e \) and \( B_s \) given by (7.16) yields the global equilibrium, and the relation of \( r_0 \) to \( J \) in (7.23) still applies, since as \( h \) tends to infinity, \( r_0/h \) goes to a constant which is much smaller than unity. This constant is around 0.007 for the case of \( r_{00} = 0.01 \). Thus, we have

\[
\frac{2s^3}{1 - s^2} - 2s + \ln \frac{1 + s}{1 - s} = \ln \frac{8Jh}{r_{00}} - 1. \tag{7.42}
\]

Then, using the frozen-flux condition at the surface of flux-rope leads to:

\[
h_0 A_R^2 \frac{I_0}{c} = h A_R
\]

\[
= 2h \left[ \frac{I}{c} \ln \frac{8Jh}{r_{00}} + \frac{s A_0 R_0}{\pi q} \tan^{-1} \left( \frac{s}{\sqrt{1 - s^2}} \right) - \frac{I}{c} \left( \ln \frac{1 + s}{1 - s} - 2s \right) \right], \tag{7.43}
\]

where both the constants \( h_0 \) and \( A_R^2 \) are the same as before. Using (7.41) and (7.42) to eliminate \( A_0 R_0 / \pi q \) and the logarithmic term in (7.43), respectively, gives

\[
A_R^2 h_0 = 2Jh \left[ 1 + \frac{2s^3}{1 - s^2} + \frac{2s^2}{\sqrt{1 - s^2}} \tan^{-1} \left( \frac{s}{\sqrt{1 - s^2}} \right) \right]. \tag{7.44}
\]

Substituting (7.44) and \( A_0 = I_0 \sigma u/c \) into (7.41), we find

\[
\sigma = \frac{\pi s^2 A_R^2 h_0}{u \sqrt{1 - s^2}} \left[ 1 + \frac{2s^3}{1 - s^2} + \frac{2s^2}{\sqrt{1 - s^2}} \tan^{-1} \left( \frac{s}{\sqrt{1 - s^2}} \right) \right]^{-1}. \tag{7.45}
\]

Equations (7.42) and (7.44) determine \( s \) and \( Jh \) uniquely for a given \( r_{00} \), reducing \( s \) and \( Jh \) to constants. Thus, according to (7.45), \( \sigma \) is also a constant. Plotting \( h \) versus \( \sigma \) gives
the vertical line in Figure 7-2, which is the limit of the open field state at $\sigma = 0.741$ for $r_{00} = 0.01$. Therefore, an open field configuration can be reached by reducing the attractive force of the Sun's dipole field below this critical value. This behavior is analogous to that found by Mikić and Linker [1994] for the opening of an arcade system when the shear at the footpoints exceeds a particular value. Due to the lack of a general solution for the configuration with a current sheet of arbitrary size, the complete $h-\sigma$ relation remains unknown (The dashed curve in Figure 7-2 is a hypothetical solution).

For any value of $\sigma$ greater than the value giving an open field, the force at infinity in the asymptotic solution is always downward. This implies that a loss of ideal-MHD equilibrium can never create an open field (at least in the small radius approximation), and this result holds true for any photospheric boundary condition since the asymptotic solution is independent of the form of the photospheric field.

### 7.5 Conclusions

Our investigation shows that the effect of large scale curvature in flux-rope models of coronal mass ejections is to reverse the relation between the radius of the flux rope and the total amount of stored magnetic energy. In the previous two-dimensional models with infinitely long, straight flux ropes, the energy decreases with increasing radius. However, for the finite length, curved flux rope discussed in this chapter, the stored energy at first increases, but then decreases. This change in behavior is caused by the fact that the curvature force becomes infinite as the radius tends to zero, and equilibrium cannot be sustained unless the flux-rope current is reduced as the radius decreases. The result of reducing the current is to lower the magnetic energy stored in the configuration.

The change in the functional behavior with radius has important consequences for the
conditions required to trigger an eruption. In both straight and curved flux-rope models, eruption does not occur unless the stored energy exceeds a threshold value. Although we are not able to determine the exact value of the threshold energy for the curved flux rope case, we are able to show that it is less than 1.53 times the coronal potential energy and that eruption still occurs for any radius greater than $2.5 \times 10^{-3}$ solar radii. Therefore, when curvature is important, configurations with large flux ropes are more likely to produce an energetic eruption than configuration with small ones.

For the specific case of a Sun-centered dipole, the ratio of the total magnetic energy to the potential magnetic energy ($W_{\text{total}}/W_{\text{potential}}$) never exceeds 1.53. Since this value is less than the ratio 1.662 for the open field, a loss of equilibrium cannot fully open the field to infinity. This result suggests that even the models which have some field lines disconnected from the boundary may still be subject to the restrictions discussed by Aly [1991] and Sturrock [1991]. The main consequence of a loss of equilibrium is that it creates a current sheet where rapid reconnection can take place. With rapid reconnection it is possible for the flux-rope to escape to infinity and for all of the magnetic energy stored in the corona to be released [Lin and Forbes 2000].

In addition to the specific case of a Sun-centered dipole, we have also analyzed equilibria with a current sheet in the limit that the flux rope is far from the Sun. This asymptotic analysis, which is valid for quite general boundary conditions, shows explicitly that at large distance the attractive force of the current sheet grows more rapidly than the repulsive force caused by the curvature of the flux rope. Thus, for any boundary conditions with our geometry, a loss of ideal-MHD equilibrium never causes the field to become fully opened. However, as the strength of the normal field at the boundary decreases, the configuration approaches the open state without ever experiencing a loss of equilibrium, and small changes
in the photospheric field strength produce large changes in the equilibrium height (see Figure 7-2). This behavior suggests that even if no loss of ideal-MHD equilibrium occurs, some type of eruptive behavior involving reconnection is likely as the photospheric field weakens.

Several important issues still remain to be investigated before the usefulness of the flux-rope models can be satisfactorily determined. First of all, there is the unknown effect of anchoring the ends of the flux rope in the photosphere. Although anchoring the ends might make an eruption more difficult, it is not clear that it would prevent a loss of equilibrium from occurring. Second, there is the question of how the two-dimensional equilibria will be affected by the kink instability. Such instabilities are inherently three-dimensional and will always occur, even when all the field lines are anchored to the solar surface, if the flux-rope is sufficiently twisted [Hood 1990]. Finally, there should be a smooth continuum of solutions between flux-rope and arcade models since an arcade is equivalent to a flux-rope with its toroidal axis below the surface of the Sun. Current observations suggest that if flux ropes do exist prior to eruption of CMEs, they are nested with an extensive arcade system from which they are not easily distinguished [Koutchmy 1997]. All of these issues will be difficult to resolve unless progress can be made in developing three-dimensional models.
Chapter 8

Summary

This thesis presents four closely related models for solar eruptions that emphasize different aspects of the phenomena. All the models are based on the idea that eruptions are produced when a magnetic flux rope in the solar corona loses its equilibrium. The first two models (Chapters 4 and 5) address the effect of reconnection on the eruption, while the last two models (Chapters 6 and 7) address the effects of emerging flux and the large scale curvature of the flux rope.

Our theoretical work in this thesis has its origin in the circuit theory of eruptions first proposed by Van Tend and Kuperus [1978] and Van Tend [1979], and later extended by Kaastra [1985], Molodenskii and Filippov [1987], and Martens and Kuin [1989]. However, following the earlier work of Forbes and Isenberg [1991], and Isenberg et al. [1993], we have either eliminated the circuit as parts of those models or replaced them by MHD principles. This is desired since the circuit concept often provided highly misleading results when applied to plasmas. As in the previous models, our models here have a flux rope which is nested within an arcade of magnetic loops. Prior to an eruption, the flux rope floats in the corona under the balance between magnetic compression and magnetic tension forces, but after an eruption occurs, magnetic compression drives the flux rope upward. The compression occurs in the field between the flux rope and the surface, and if the flux rope is curved, it also occurs at the internal edge of the flux rope where the poloidal field...
is pinched by the curvature.

Within the framework of the catastrophe theory developed by Thom [1972], we have investigated the effects of magnetic reconnection, new emerging flux and large-scale curvature on the evolution, and energetics of the flux rope before and after the loss of ideal-MHD equilibrium. What we have learned about each of these aspects is summarized in the following sections.

8.1 Magnetic Reconnection

We started by re-investigating the model of Forbes and Isenberg [1991]. They used a two-dimensional dipole to describe the photospheric field in the absence of the flux rope, but when the flux rope is added it is possible for an X-line to appear in the pre-eruptive state. Forbes and Isenberg [1991] assumed that no reconnection occurred in the pre-eruptive state, so that when an X-line appears at the boundary, it immediately led to the formation of a current sheet. However, the assumption that no reconnection occurs prior to eruption is not thought any more to be appropriate for the corona since a variety of observations indicate that the reconnection time scale is short compared to the photospheric time scale.

In our work here, we have re-examined how their model works when the assumption is made that the reconnection rate is much faster than the rate at which the photosphere evolves. Although much of the behavior is the same as before, the most remarkable difference is that there is no longer a critical radius needed for an eruption to occur. In the original model of Forbes and Isenberg [1991], there was always a small current sheet prior to the loss of equilibrium, but in our model there is no such current sheet but only an X-line. Because of the absence of this current sheet, the field line tension associated with the current sheet is eliminated, so the system behaves initially just as in the original Van Tend and Kuperus
and Martens and Kuin models. Therefore, when the reconnection rate is faster than the photospheric evolutionary rate, we do not need to worry about the critical radius as none exists. Thus, in contrast to the original conclusions of Forbes and Isenberg [1991], we now find that eruptive behavior can occur with the dipole boundary condition for quite plausible values of the flux rope radius.

As soon as the catastrophic loss of equilibrium occurs in any of the flux rope models, the system evolves dynamically at the Alfvén time scale. The evolution is consequently so fast that a long current sheet quickly forms before reconnection can dissipate it. The associated magnetic tension becomes strong enough to prevent the flux rope from escaping, so in our two-dimensional models, magnetic reconnection is essential for escape.

Using a self-consistent procedure for coupling the eruption dynamics to the reconnection process, we have determined that quite small rates of reconnection are sufficient to make escape possible. Specifically, for an Alfvén Mach number, \( M_A > 0.005 \), in the inflow of reconnection region, quite reasonable trajectories are obtained. Since the values of \( M_A \) inferred from observations are greater than this value, our results show that the two-dimensional flux rope model yields results which are consistent with the observations.

On the other hand, because the time scale of magnetic reconnection is much longer than the Alfvén time scale, an extensive current sheet still forms after the catastrophe even when \( M_A \) is of order unity. Only during the late phase several hours after onset, does the current sheet start to shorten. For values of \( M_A \) less than unity, the lower tip of the current sheet rises very slowly during the early phase after onset, while the upper tip rises very quickly at near the same speed as the flux rope itself. Thus, the field becomes highly extended before a substantial amount of reconnection has occurred. This behavior is consistent with that observed by the LASCO coronagraph on SOHO. The best fit to observations is obtained by
taking $M_a \approx 0.1$, with which the energy output matches the temporal behavior inferred for the long duration events often associated with CMEs and other major eruptions. There is a relation between the output power of the eruption and the growth and decay of the current sheet. The quick drop of output power at the end of the impulsive phase is related to the development of a long current sheet, while the long lifetime of the late phase is related to the extended time it takes for reconnection to convert the free energy left in the system.

The model also provided an explanation for the peculiar motion of giant X-ray arches reported by Švestka et al. [1995 and 1997]. These giant arches are similar in form to “post”-flare loops, but are associated with CMEs and have an upward motion pattern that is different than that of “post”-flare loops. The model predicts that at lower altitudes of “post”-flare loops, the rise of the loops should continually slow with time, and that at higher altitude of giant arches, the rise rate of the arches should increase with time. The transition between the two different patterns of motion happens when the neutral point reaches the altitude where the local Alfvén speed starts to increase with height due to the fall off in coronal plasma density.

### 8.2 Newly Emerging Flux

As one of the most important processes of transporting energy into the coronal field, the emergence of new magnetic flux from below the photosphere plays an essential role in evolving coronal magnetic structures and in triggering eruptions. We used a simple model to investigate how an existing magnetic configuration containing a flux rope changes in response to newly emerging flux, and we looked for a simple possible relation between the eruption and the orientation of the new emerging flux region relative to a pre-existing flux region, but we did not find such a relation.
The emergence of new flux was modeled by varying various parameters which characterize the new emerging flux region. We found that it is possible to create an imbalance to cause the flux rope to be ejected upward. Altogether, there are four free parameters to prescribe the boundary condition (strengths of old and new flux systems, and horizontal and vertical locations of new flux), and one free parameter to prescribe the flux rope (i.e. its radius). This large of number of free parameters yielded a multi-dimensional equilibrium surface with complex geometry and numerous different scenarios which one could construct to have an eruption. Therefore, it is almost impossible to extract a simple rule for predicting what types of emerging flux would produce an eruption.

So, the simple picture that new flux with an orientation favorable for reconnection will trigger an eruption by the so-called tether-cutting (see Canfield et al. [1994]) of the field lines overlying the flux rope is insufficient to describe the behavior we observed. For a given configuration, a loss of equilibrium does not necessarily occur just because of a decrease in the magnetic tension or an increase in the magnetic compression. Only at special locations, namely critical points, can force balance be lost and catastrophe take place, but these location are difficult to determine by only using a simple cartoon picture. Magnetic reconnection alone is not guaranteed to drive the system to one of these critical points.

8.3 Large-Scale Curvature

To construct a more realistic model of CMEs, we introduced the large scale curvature, which any realistic flux rope must have. To do this, we replaced the flat solar surface and an infinitely long cylindrical flux rope of the previous models with a spherical solar surface and a circular flux rope around the Sun. Although this model is still geometrically
oversimplified compared to real CMEs, it contains two important improvements over the old
two-dimensional models. First, it includes the curvature force that is almost certain to be
important for driving the CME outward from the Sun, and second, the total energy stored
in the system is finite instead of infinite as in the two-dimensional models with translational
symmetry. As expected, we found the large-scale curvature does indeed help propel the flux
rope outward from the Sun.

The finite scale of the curved configuration allows us to calculate the total energy stored
in the system prior to eruption. In our specific case of a Sun-centered dipole, the ratio of
the total magnetic energy to the potential magnetic energy does not exceed 1.53 which is
less than the value of 1.662 for the open field. Thus, the loss of ideal-MHD equilibrium does
not fully open the field to infinity, even though the model has some field lines disconnected
from the boundary and is, therefore, not subject to the Aly-Sturrock constraint. As in
the Cartesian case, the main consequence of a loss of ideal-MHD equilibrium is to create
a current sheet where rapid reconnection can occur so that the flux rope can escape into
interplanetary space.

We also show, using an asymptotic analysis, that when the flux rope is very far from the
Sun, the attractive force of the current sheet grows more rapidly than the repulsive force
causd by the curvature of the flux rope. Thus an ideal-MHD loss of equilibrium never
causes the field to fully open up.

8.4 Conclusions

This thesis has concentrated on analytic models of solar eruptions based on the magnetic
configuration of a current carrying flux rope nested within an arcade of magnetic loops. The
evolution of these models is based on fundamental characteristics of the solar corona, such
as force-free and frozen-in conditions, and the photosphere, such as magnetic line-tying. By focusing on the key factors of the system, our models highlight the most essential features of the system.

Our analytic models allow us to investigate thoroughly the response of the system to the variation of the parameters which specify the background condition, and they also provide us with precise information, such as the locations of critical points and how to evolve the boundary condition so as to reach these points. Thus, our analytical solutions provide information that is difficult to extract from numerical simulations. Since analytic solutions also allow us to follow the evolution process as long as we want, we have been able to calculate for the first time from a specific model, the long-term behavior of the eruption. Specifically, we have determined the position of the flux rope, the magnetic output power, the length of current sheet, and the electrical field at the reconnecting current sheet as functions of time.

Our analytic model of the effect of emerging flux on flux-rope equilibrium has established that the eruption behavior is potentially far more complex than previously supposed. Although numerical simulations of emerging flux have been carried out to look for eruptive behavior, none of them has been able to obtain eruptive behavior without introducing an artificially rapid change in the system (e.g. either impulsively driving the boundary conditions or arbitrarily increasing the resistivity). This may be due, in part, to the fact that the simulations have lacked guidance as to how to vary the boundary conditions properly. Our new analytical models suggest several ways to set up the boundary conditions in the simulations so as to obtain eruptive behavior.

So far, we have only discussed what our models and analytic solution can do. However, we will not end this thesis without mentioning several important issues that still remain to
be investigated.

First of all, the flux rope in all of our models is detached from the boundary, so the effect of anchoring the ends of the flux rope in the photosphere remains unknown. Although anchoring the ends might make an eruption more difficult, it seems unlikely that it would prevent a loss of equilibrium from occurring.

Second, there is a question of how the equilibria are affected by the kink instability. Such instabilities are inherently three-dimensional and can occur, even when the ends of flux rope are anchored to the solar surface, if the flux rope is sufficiently twisted [Hood 1990]. Recently, Titov and Démoulin [1999] analyzed a special case of a circular flux rope which is imbedded in a line-tying surface. They considered the stability and argued that this configuration would become unstable if the major radius of the circular flux rope was large enough. Although they did not prove this rigorously because they neglected the effects of line-tying, they could establish that the configuration had equilibrium properties similar to the completely symmetric configuration of Lin et al. [1998].

Third, there should be a smooth continuum of solutions between flux rope and arcade models since an arcade is equivalent to a flux rope with its toroidal axis below the surface of the Sun. Current observations suggested that if flux ropes do exist prior to eruption of CMEs, they are nested with an extensive arcade system from which they are not easily distinguished [Koutchmy 1997]. A couple of recent numerical experiments have also confirmed that if the toroidal axis of a flux rope lies beneath the photospheric surface, then the part of the rope visible in the corona matches the arcades that are actually observed [Magara 2001; and Fan 2001].

Fourth, although we have already investigated the importance of magnetic reconnection to the eruptive phenomena, there is still much that remains to be determined. For example,
what is the actual dynamics of the reconnection process in the current sheet during an eruption, especially within the first hours after onset when, as our model predicts, the length of current sheet increases at almost the Alfvén speed.

Finally, a great deal of work is needed to determine the effects of shocks, reconnection heating, and the solar wind on the eruptive process. Hopefully, more realistic models which include these effects will be developed by the solar research community in the near future.
Bibliography


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Appendix A

Complex Notation for $A(x, y)$

The flux function $A(x, y)$ in equation (5.2) is the real part of the complex flux function $A(z)$ as

$$A(z) = \frac{i_0}{c} \left[ 2J \ln \left( \frac{z + ih}{z - ih} \right) + i \ln \left( \frac{z + \lambda}{z - \lambda} \right) + \frac{2}{\pi} \int_{\rho}^q \ln \left( \frac{z + iv}{z - iv} \right) \tilde{K}_s(v) dv \right], \quad (A.1)$$

where $\tilde{K}_s(v)$ is given by (5.5) or, equivalently

$$\tilde{K}_s(v) = \frac{\lambda}{\pi} \frac{(h^2 + \lambda^2) \sqrt{(v^2 - p^2)(q^2 - v^2)}}{\sqrt{(\lambda^2 + p^2)(\lambda^2 + q^2)(2v^2 - \lambda^2)}} \quad p \leq v \leq q. \quad (A.2)$$

Here, the Y-point conditions [(5.7), (5.8), and (5.9)] at both tips of the current sheet have been used to deduce (A.2).

In principle, the complex flux function $A(z)$ can be obtained by substituting $\tilde{K}_s(v)$ into (A.1) and completing the integral, but the algebra is rather length. Instead, we use the general relation between $A(z)$ and $B(z)$, namely

$$B(z) = -\frac{\partial A(z)}{\partial z} \quad (A.3)$$

to calculate $B(z)$ first, and then integrate $B(z)$ over $z$ to obtain $A(z)$.

Combining equations (A.2), (A.1), and (A.3), and then completing the integral, we find

$$B(z) = \frac{2i_0}{c} \frac{1}{\sqrt{(z^2 + p^2)(z^2 + q^2)}} \left[ 2Jh + \lambda + P_0 - \frac{2Jh \sqrt{(h^2 - p^2)(h^2 - q^2)}}{z^2 + h^2} \right] - \frac{\lambda \sqrt{(\lambda^2 + p^2)(\lambda^2 + q^2)}}{z^2 - \lambda^2}.$$  

This expression can be further simplified by using the Y-point conditions (5.7), (5.8), and (5.9) to obtain

$$B(z) = \frac{2i_0}{c} \frac{\lambda(h^2 + \lambda^2) \sqrt{(z^2 + p^2)(z^2 + q^2)}}{\sqrt{(\lambda^2 + p^2)(\lambda^2 + q^2)(z^2 - \lambda^2)(z^2 + h^2)}}, \quad (A.4)$$

which is equation (5.13). Finally, from equations (A.3) and (A.4), we have

$$A(z) = -\int B(z) dz$$

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\[ \frac{2I_0}{c} \frac{1}{(q-p)\sqrt{(\lambda^2+p^2)(\lambda^2+q^2)}} \left( 2i\lambda(\lambda^2+q^2) \right) \]

\[ \cdot F\left\{ \sin^{-1}\left[ \frac{(p-q)(p-i\lambda)}{(p+q)(p+i\lambda)} \right], \frac{p+q}{p-q} \right\} \]

\[ + \frac{\lambda}{q(p-q)(\lambda^2+h^2)} F\left[ i\sinh^{-1}\left( \frac{z}{\lambda} \right), \frac{p}{q} \right] \]

\[ + \frac{\lambda}{h^2q(p-q)(h^2-p^2)(h^2-q^2)} \Pi \left[ i\sinh^{-1}\left( \frac{z}{\lambda} \right), \frac{p^2}{h^2}, \frac{p}{q} \right] \]

\[ - 2p(\lambda^2+p^2) \Pi \left\{ \sin^{-1}\left[ \frac{(p-q)(p-i\lambda)}{(p+q)(p+i\lambda)} \right], \frac{(p+q)(p-i\lambda)}{(p-q)(p+i\lambda)}, \frac{p+q}{p-q} \right\} \]

\[ + 2p(\lambda^2+p^2) \Pi \left\{ \sin^{-1}\left[ \frac{(p-q)(p-i\lambda)}{(p+q)(p+i\lambda)} \right], \frac{(p+q)(p+i\lambda)}{(p-q)(p-i\lambda)}, \frac{p+q}{p-q} \right\} \], \text{(A.5)}

which is equation (5.14).
Appendix B

Derivation of Poisson’s Equation for $A_\phi$

To derive equation (7.7) for the vector potential component $A_\phi$, we start with Ampère’s law:

$$\nabla \times \mathbf{B} = \frac{4\pi}{c} \mathbf{J}, \quad (B.1)$$

and substitute $\mathbf{B} = \nabla \times \mathbf{A}$ which leads to

$$\nabla \times (\nabla \times \mathbf{A}) = -\nabla^2 \mathbf{A} = \frac{4\pi}{c} \mathbf{J},$$

upon applying the Coulomb gauge $\nabla \cdot \mathbf{A} = 0$. Thus, for the $\phi$ component, we have

$$[\nabla^2 \mathbf{A}]_\phi = -\frac{4\pi}{c} J_\phi. \quad (B.2)$$

Since the unit vectors in spherical coordinates are functions of position, the left hand side of (B.2) is

$$[\nabla^2 \mathbf{A}]_\phi = \nabla^2 A_\phi - \frac{A_\phi}{r^2 \sin^2 \theta} \quad (B.3)$$

(See page 116 of Morse and Feshbach [1953]). Substituting (B.3) into (B.2) yields

$$\nabla^2 A_\phi - \frac{A_\phi}{r^2 \sin^2 \theta} = -\frac{4\pi}{c} J_\phi$$

which is equivalent to

$$\nabla^2 (A_\phi \cos \phi) = -\frac{4\pi}{c} J_\phi \cos \phi, \quad (B.4)$$

which is equation (7.7).
Appendix C

Determination of The Shape Factor in Shafranov's Equation

Shafranov's expression for the external magnetic field, $B_s$, needed to maintain equilibrium in a thin, force-free flux rope is

$$B_s = \frac{I}{ch} \left( \ln \frac{8h}{r_0} - \frac{3}{2} + \frac{l_i}{2} \right), \quad (C.1)$$

where $l_i$ is the internal self-inductance per unit length of the flux-rope. This inductance, which is associated with the toroidal current density $j_z$ (in the coordinate system described in section 7.3.2), depends on the shape-of the current density distribution inside the flux rope. In most applications the current density is assumed to be uniform inside the flux rope which gives $l_i = 1/2$. However, in our case, the current density is prescribed by the Lundquist [1950] solution (7.19) for which

$$j_z = \frac{c}{4\pi} \lambda B_0 J_0(\lambda r). \quad (C.2)$$

According to Shafranov [1966], the general formula for $l_i$ is

$$l_i = \frac{c^2 W_z}{\pi h I^2}, \quad (C.3)$$

where

$$I = 2\pi \int_0^{r_0} j_z r dr \quad (C.4)$$

and

$$W_z = \frac{1}{8\pi} \int_V B_z^2 dV, \quad (C.5)$$

here the volume integral in (C.5) is taken over the whole volume of the flux-rope. Substituting (C.2) into (C.4) and (7.19) into (C.5) yields

$$l_i = 1, \quad (C.6)$$

so, (C.1) becomes

$$B_s = \frac{I}{ch} \left( \ln \frac{8h}{r_0} - 1 \right). \quad (C.7)$$