Fall 2001

Reflexivity, elementary operators and cohomology

Jiankui Li
University of New Hampshire, Durham

REFLEXIVITY, ELEMENTARY OPERATORS AND COHOMOLOGY

By

Jiankui Li

M.S. Qufu Normal University (1987)

DISSERTATION

Submitted to the University of New Hampshire
in partial fulfillment of
the requirements for the degree of

Doctor of Philosophy
in
Mathematics

September 2001

UMI Number: 3022960

This dissertation has been examined and approved.

Director, Don Hadwin
Professor of Mathematics

Eric Nordgren
Professor of Mathematics

Rita Hibschweiler
Professor of Mathematics

Edward Hinson
Associate Professor of Mathematics

Ken Harrison
Associate Professor of Mathematics
Murdoch University

6/27/01
Date

Dedication

To my family.

Acknowledgments

I would like to thank my advisor, Don Hadwin for his guidance and support, encouragement.

I would like to thank Rita Hibschweiler for her help during the preparation this paper and when writing up the paper. She spends a lot time to read carefully the paper.

I would like to thank to Eric Nordegren. I have learned a lot from him.

I want to thank Edward Hinson and Ken Harrison; the other members of my Dissertation Committee.

I also whish to thank my friend Dr. Z. Pan for the successful cooperation.

Last but definitely not the least, I would like to thank my friends Hemant Pendharkar and Jeff Zak for their help.

CONTENTS

Abstract

Reflexivity, elementary operators and cohomology

by

Jiankui Li

University of New Hampshire, September 2001

Let H be a separable complex Hilbert space and let $B(H)$ be the set of all bounded operators on H. In this dissertation, we show that if $\mathcal{S}$ is a n-dimensional subspace of $B(H)$, then $\mathcal{S}$ is $[\sqrt{2n}]$-reflexive, where $[t]$ denotes the largest integer that is less than or equal to t.

We obtain some lattice-theoretic conditions on a subspace lattice $\mathcal{L}$ which imply $alg\mathcal{L}$ is strongly rank decomposable. Let $\mathcal{S}$ be either a reflexive subspace or a bimodule of a reflexive algebra. We find some conditions such that T has a rank one summand in $\mathcal{S}$ and $\mathcal{S}$ has strong rank decomposability. Let $\mathcal{S}(\mathcal{L})$ be the set of all operators on H that annihilate all the operators of rank at most one in $alg\mathcal{L}$. Katavolos, Katsoulis and Longstaff show that if $\mathcal{L}$ is a subspace lattice generated by two atoms, then $\mathcal{S}(\mathcal{L})$ is strongly rank decomposable. They ask whether $\mathcal{S}(\mathcal{L})$ is strongly rank decomposable if $\mathcal{L}$ is an atomic Boolean subspace latttice with more than two atoms. For any $n \geq 3$, we construct an atomic Boolean subspace lattice $\mathcal{L}$ on H with n atoms such that there is a finite rank operator T in $\mathcal{S}(\mathcal{L})$ such that T does not have a rank one summand in $\mathcal{S}(\mathcal{L})$. This answers their question negatively. We also discuss isomorphisms of reflexive algebras.

We introduce a new concept called "bounded reflexivity" for a subspace of operators on a normed space. We explore the properties of bounded reflexivity, and we compare the similarities and differences between bounded reflexivity and the usual reflexivity for a subspace of operators. We discuss the relations of bounded reflexivity of subspaces of $B(H)$ and complete positivity of elementary operators on $B(H)$. As applications of bounded reflexivity, we give shorter proofs of some well known results about positivity and complete positivity of elementary operators. By using those ideas, we study properties of a C^*-algebra in which every n-positive elementary operator is completely positive. We study the derivations in nonselfadjoint algebras. We research derivations on a nest subalgebra of von Neumann algebras. We also consider two cohomology theories, the norm continuous cohomology and the normal cohomology on some nonselfadjoint algebras. Those algebras contain reflexive algebras whose invariant subspace lattices are tensor products of nests and reflexive algebras whose invariant subspace lattices are generated by two atoms. We obtain for those algebras $\mathcal{A}$ that $H_c^n(\mathcal{A}, B(H)) = H_w^n(\mathcal{A}, B(H))$.

Introduction

Operator algebras originated in the work of von Neumann, in particular in his search for a natural mathematical frame for Quantum Mechanics, and in the work of Gelfand and Naimark. Operator algebras can be viewed as a discipline encompassing Noncommutative Analysis, Geometry and Topology. Operator algebras are undoubtedly one of the mathematical fields most notable for the depth of the problems, the richness of new ideas, and the numerous connections to a variety of other fields. In addition the field offers great potential as a unifying language and as a source of illumination, helping to explain other problems and providing a framework for further research.

In the 1960s, because the theory of selfadjoint operators and operator algebras had undergone a vigorous and moderately successful development, people began to investigate how far the theorems of selfadjoint theory could be generalized and what forms they should take in the new context. In [54] Kadison and Singer introduced triangular operator algebras. Non-selfadjoint operator algebras really began with their pioneering paper "Triangular operator algebras". Nest algebras were introduced by Ringrose as generalizations of certain triangular algebras. Now nest algebras play an important role in non-selfadjoint algebras (see [20]). It was Ringrose's proof that complete nests are reflexive that was the starting point for Halmos's introduction of reflexive algebras and lattices. We can consider that the non-selfadjoint generalizations of von Neumann algebras are the reflexive algebras. The notion of reflexivity was first introduced by Halmos in 1971 for subalgebras of algebra

$B(H)$. Loginov and Sulman [79] extend reflexivity to include subspaces $B(H)$ which are not necessarily algebras.

Now we introduce some basic notation and some definitions. Standard terminology and notation will be used. The Hilbert spaces which we consider are all complex and separable. The terms *operator* and *subspace* mean *bounded* operator and *closed* subspace respectively. We denote by $B(H)$ the set of all operators on H, $K(H)$ the set of all compact operators on H and $F(H)$ the set of all finite rank operators on H. For any subset $\mathcal{S}$ of $B(H)$, define $\mathcal{S}^{(n)} = \{S^{(n)} \in B(H^{(n)}) : S \in \mathcal{S}\}$, where $H^{(n)}$ is the direct sum of n copies of H and $S^{(n)}$ is the direct sum of n copies of S acting on $H^{(n)}$. For x, y in H, let $x \otimes y$ denote the rank-one operator $u \mapsto (u, y)x$, whose norm is $||x||||y||$. We let $\mathcal{S}^* = \{S^* : S \in \mathcal{S}\}$. In this paper, " $\subseteq$ " is used for set inclusion while " $\subset$ " is reserved for proper inclusion. For convenience we disregard the distinction between a subspace of H and the orthogonal projection on it.

This dissertation contains three chapters. In Chapter One, we consider reflexivity of subspaces of $B(H)$, strong rank decomposability of reflexive algebras and bimodules of reflexive algebras and algebraic isomorphisms of some reflexive algebras. In section 1.1, in collaboration Z. Pan, the main result is Theorem 1.13. This Theorem answers a question of Magajna [85]. In section 1.2, we obtain some lattice-theoretic conditions on a subspace lattice $\mathcal{L}$ which imply $alg\mathcal{L}$ is strongly rank decomposable. Let $\mathcal{S}$ be either a reflexive subspace or a bimodule of a reflexive algebra. We find some conditions such that T has a rank one summand in $\mathcal{S}$ and $\mathcal{S}$ has strong rank decomposability. Let $\mathcal{S}(\mathcal{L})$ be the set of all operators on H that annihilate all the operators of rank at most one in $alg\mathcal{L}$. In [56], Katavolos, Katsoulis and Longstaff show that if $\mathcal{L}$ is a subspace lattice generated by two atoms, then $\mathcal{S}(\mathcal{L})$ is strongly rank decomposable. For $n \geq 3$, we construct an atomic Boolean subspace lattice $\mathcal{L}$ on H with n atoms such that there is a finite rank operator T in $\mathcal{S}(\mathcal{L})$ such that T does not have a rank one summand in $\mathcal{S}(\mathcal{L})$. This answers their a question in [56] negatively. In section 1.3, we discuss isomorphisms of reflexive algebras.

Chapter 2 studies bounded reflexivity and applications. In section 2.1, together with Z. Pan, we introduce a new concept "bounded reflexivity" for a subspace of operators on a normed space. We explore the properties of bounded reflexivity, study the similarities and differences between bounded reflexivity and the usual reflexivity for a subspace of operators. In section 2.2, we discuss the relation between bounded reflexivity of subspaces of $B(H)$ and complete positivity of elementary operators on $B(H)$. As applications of bounded reflexivity, we give shorter proofs of some well known results about positivity and complete positivity of elementary operators. In section 2.3, we use the ideas in sections 2.1 and 2.2, to study the properties of a C^*-algebra on which every n-positive elementary operator is completely positive.

In [99], Sakai proves that if $\mathcal{A}$ is a von Neumann algebra, then $H^1_c(\mathcal{A}, \mathcal{A}) = 0$. It is an open question whether for any von Neumann algebra $\mathcal{A}$, $H^n_c(\mathcal{A}, \mathcal{A}) = 0$. For non-selfadjoint algebras, in [64], Lance shows that if $\mathcal{A}$ is a nest algebra then $H^n_c(\mathcal{A}, B(H)) = 0$.

In the last chapter, we unify some results on derivations by considering derivations from an algebra $\mathcal{A}$ containing all rank one operators of a nest algebra into an $\mathcal{A}$-bimodule $\mathcal{B}$. We study derivations on nest subalgebra of von Neumann algebras. We also consider two cohomology theories, the norm continuous cohomology and the normal cohomology on some nonselfadjoint algebras. These algebras contain reflexive algebras whose invariant subspace lattices are tensor products of nests and reflexive algebras whose invariant subspace lattices are generated by two atoms.

Chapter 1

Reflexive Subspaces and Rank Decomposability

Let H be a complex separable Hilbert space. For any set $\mathcal{F}$ of subspaces of H, we define

$$alg\mathcal{F} = \{T \in B(H) : TM \subseteq M, \text{ for any } M \in \mathcal{F}\}.$$

Obviously for any collection $\mathcal{F}$ of subspaces, $alg\mathcal{F}$ is a weakly closed subalgebra of $B(H)$ containing I.

For any subset $\mathcal{A}$ of $B(H)$, the set of invariant subspaces of $\mathcal{A}$ is denoted by $lat\mathcal{A}$. Thus

$$lat\mathcal{A} = \{M : TM \subseteq M, \text{ for any } T \in \mathcal{A}, M \text{ is a subspace of } H\}.$$

Let $\mathcal{A}$ be a subalgebra of $B(H)$. Obviously $\mathcal{A} \subseteq alglat\mathcal{A}$. We say that $\mathcal{A}$ is reflexive if $\mathcal{A} = alglat\mathcal{A}$.

For any subspace $\mathcal{S} \subseteq B(H)$, define $ref(\mathcal{S}) = \{T \in B(H) : Tx \in [\mathcal{S}x], \text{ for any } x \in H\}$, where $[\cdot]$ denotes norm closed linear span. $\mathcal{S}$ is called reflexive if $ref(\mathcal{S}) = \mathcal{S}$. $\mathcal{S}$ is called n-reflexive if $\mathcal{S}^{(n)}$ is reflexive in $B(H^{(n)})$. If $\mathcal{A}$ is a subalgebra of $B(H)$ containing I, then $\mathcal{A}$ is reflexive as an algebra if and only if $\mathcal{A}$ is reflexive as a subspace of $B(H)$.

1.1 Reflexivity of finite dimensional subspaces

Let $\mathcal{S}$ be a subspace of $B(H)$. A vector $x \in H$ is called a separating vector of $\mathcal{S}$ if the map $E_x : S \to Sx$, $S \in \mathcal{S}$ is injective. Let $sep(\mathcal{S})$ denote the set of all separating vectors of $\mathcal{S}$ in H. The local dimension of $\mathcal{S}$, denoted by $k(\mathcal{S})$, is defined by $k(\mathcal{S}) = max\{dim\,[Sx : S \in \mathcal{S}] : x \in H\}$. It is clear that $k(\mathcal{S}) \leq dim\,\mathcal{S}$. If $dim\,\mathcal{S} < \infty$, it is not hard to see that $sep(\mathcal{S}) \neq \emptyset$ if and only if $k(\mathcal{S}) = dim\,\mathcal{S}$. N-reflexivity of a subspace of $B(H)$ has been considered, for example, in [8, 61]. In [67], Larson proved that if $\mathcal{S}$ is a finite dimensional subspace of $B(H)$, then $ref(\mathcal{S}^{(n)}) = \mathcal{S}^{(n)} + ref(\mathcal{S}^{(n)} \cap F(H^{(n)}))$. It follows immediately that $\mathcal{S}$ is n-reflexive if and only if $\mathcal{S} \cap F(H)$ is n-reflexive. Hence we are only interested in which finite dimensional subspaces of $F(H)$ are n-reflexive. In [70], we show that if $\mathcal{S}$ is an n-dimensional subspace of $B(H)$, then $\mathcal{S}$ is $([\frac{n}{2}]+1)$-reflexive. In this section, our main result is Theorem 1.13. Theorem 1.13 proves that if $\mathcal{S}$ is an n-dimensional subspace of $B(H)$, then $\mathcal{S}$ is $[\sqrt{2n}\,]$-reflexive. Example 1.14 shows that $[\sqrt{2n}\,]$ is the smallest integer such that all n-dimensional subspaces of $B(H)$ are $[\sqrt{2n}\,]$-reflexive.

In the following, we always assume that $\mathcal{S}$ is a subspace of $B(H)$, $dim\ \mathcal{S} < \infty$, and $\mathcal{S} \subseteq F(H)$ unless stated otherwise. Before we prove our main result, we need several lemmas and propositions.

Lemma 1.1[39]. *The set $sep(\mathcal{S})$ is an open subset of H.*

Lemma 1.2[39]. *The set $sep(\mathcal{S})$ is either empty or dense in H.*

Let M be a closed subspace of H and P be the orthogonal projection of H onto M. Define $\mathcal{S}_M = \{S \in \mathcal{S} : R(S) \subseteq M\}$, where $R(S)$ is the range of S. Let $\mathcal{S}_M^c$ be any vector space complement of $\mathcal{S}_M$ in $\mathcal{S}$. Define $P^\perp\mathcal{S}_M^c = \{P^\perp S : S \in \mathcal{S}_M^c\}$.

Proposition 1.3. $k(\mathcal{S}_M) + k(P^\perp \mathcal{S}_M^c) \leq k(\mathcal{S})$.

Proof. If $P^\perp\mathcal{S}_M^c = 0$, it is obvious that $k(\mathcal{S}_M) \leq k(\mathcal{S})$.

If $\mathcal{S}_M = 0$, it follows that $\mathcal{S}_M^c = \mathcal{S}$ and

$$k(P^\perp \mathcal{S}_M^c) = max\{dim\ [P^\perp Sx : S \in \mathcal{S}_M^c] :\ x \in H\} \leq max\{dim\ [Sx] :\ x \in H\} = k(\mathcal{S}).$$

Now suppose $k(\mathcal{S}_M) = m \neq 0$ and $k(P^\perp \mathcal{S}_M^c) = l \neq 0$. Let $x_0 \in H$ be a separating vector of $span\{S_1, ..., S_m\} \subseteq \mathcal{S}_M$. Similarly there exist $P^\perp T_1, ..., P^\perp T_l \in \mathcal{S}_M^c$ such that $span\{P^\perp T_1, ..., P^\perp T_l\}$ has a separating vector. By Lemmas 1.1 and 1.2, we can choose $y \in H$ with $||y||$ small enough so that $x_0 + y$ is a separating vector for $span\{S_1, ..., S_m\}$ and $span\{P^\perp T_1, ..., P^\perp T_l\}$. For any $\lambda_1, ..., \lambda_m, \mu_1, ..., \mu_l \in \mathbf{C}$, suppose

$$\lambda_1 S_1(x_0 + y) + ... + \lambda_m S_m(x_0 + y) + \mu_1 T_1(x_0 + y) + ... + \mu_l T_l(x_0 + y) = 0. \tag{1.1}$$

Applying $P^\perp$ to both sides of (1.1), it follows

$$\mu_1 P^\perp T_1(x_0 + y) + ... + \mu_l P^\perp T_l(x_0 + y) = 0. \tag{1.2}$$

Since $x_0 + y$ is a separating vector of $span\{P^\perp T_1, ..., P^\perp T_l\}$, (1.2) yields $\mu_1 = ... = \mu_l = 0$. Now (1.1) implies $\lambda_1 = ... = \lambda_m = 0$, since $x_0 + y$ is a separating vector of $span\{S_1, ..., S_m\}$. Hence $k(\mathcal{S}) \geq k(\mathcal{S}_M) + k(P^\perp \mathcal{S}_M^c)$. □

Proposition 1.4. *If $k(\mathcal{S}_M) = dim\ M$, then $k(\mathcal{S}_M) + k(P^\perp \mathcal{S}_M^c) = k(\mathcal{S})$.*

Proof. By Proposition 1.3, we only need to prove $k(\mathcal{S}) \leq k(\mathcal{S}_M) + k(P^\perp \mathcal{S}_M^c)$.

Suppose that $k(\mathcal{S}_M) = m$ and $k(P^\perp \mathcal{S}_M^c) = l$. If $m + l = dim\ \mathcal{S}$, it is obvious that $k(\mathcal{S}) \leq k(\mathcal{S}_M) + k(P^\perp \mathcal{S}_M^c)$. If $m + l < dim\ \mathcal{S}$, and $m + l < n \leq dim\ \mathcal{S}$, we take n linearly independent operators from $\mathcal{S}$ in such a way that $S_1, ..., S_{m_1} \in \mathcal{S}_M, T_1, ..., T_{l_1} \in \mathcal{S}_M^c$ and $m_1 + l_1 = n$. For any nonzero x_0 in H, we show that there are $\lambda_1, ..., \lambda_{m_1}, \mu_1, ..., \mu_{l_1}$, not all zero, such that

$$\lambda_1 S_1 x_0 + ... + \lambda_{m_1} S_{m_1} x_0 + \mu_1 T_1 x_0 + ... + \mu_{l_1} T_{l_1} x_0 = 0. \tag{1.3}$$

If $l_1 \leq l$, then $m_1 > m$, choose $\mu_1 = ... = \mu_{l_1} = 0$. Since $k(S_M\} = m$, it follows that there are $\lambda_1, ..., \lambda_{m_1}$, not all zero, such that $\lambda_1 S_1 x_0 + ... + \lambda_{m_1} S_{m_1} x_0 = 0$. Suppose that $l_1 > l$. If $span\{P^{\perp}T_1x_0, ..., P^{\perp}T_{l_1}x_0\} = (0)$, then $span\{T_1x_0, ..., T_{l_1}x_0\} \subseteq M$. Because $k(S_M) = dim\ M$, and $l_1 + m_1 = n > m + l$, it follows that there are $\lambda_1, ..., \lambda_{m_1}, \mu_1, ..., \mu_{l_1}$, not all zero, satisfying (1.3). Without loss of generality, we may assume that $\{P^{\perp}T_1x_0, ..., P^{\perp}T_tx_0\}, 1 \leq t \leq l$ is linearly independent, and $P^{\perp}T_jx_0 \in span\{P^{\perp}T_1x_0, ..., P^{\perp}T_tx_0\}, t+1 \leq j \leq l_1$. Suppose that $P^{\perp}T_jx_0 = \sum_{i=1}^{t} a_{ij}P^{\perp}T_ix_0, t+1 \leq j \leq l_1$. Let $B_j = T_j - \sum_{i=1}^{t} a_{ij}T_i$. Then $B_jx_0 \in M, t+1 \leq j \leq l_1$. Since $S_ix_0 \in M, 1 \leq i \leq m_1$ and $dim\ M = m < m_1 + l_1 - l \leq m_1 + l_1 - t$, we may choose $\lambda_1, ..., \lambda_{m_1}$ and $\mu_{t+1}, ..., \mu_{l_1}$, not all zero, such that

$$\lambda_1 S_1 x_0 + ... + \lambda_{m_1} S_{m_1} x_0 + \mu_{t+1} B_{t+1} x_0 + ... + \mu_{l_1} B_{l_1} x_0 = 0. \quad (1.4)$$

Hence

$$\lambda_1 S_1 x_0 + ... + \lambda_{m_1} S_{m_1} x_0 + \mu_{t+1}(T_{l_1} - \sum_{i=1}^{t} a_{i\ t+1} T_i)x_0 + ... + \mu_{l_1}(T_{l_1} - \sum_{i=1}^{t} a_{i\ l_1} T_i)x_0 = 0. \quad (1.5)$$

By (1.5), it follows that (1.3) is true. □

Lemma 1.5[23]. *Let V be a vector space over a field $\mathbf{F}$ and let $L(V)$ be the set of all linear transformations on V. Suppose $\mathcal{S} \subseteq L(V)$ and dim $\mathcal{S}$ is less than the cardinality of $\mathbf{F}$. Let x be a separating vector of $\mathcal{S}$ and W be a linear subspace of V satisfying $\mathcal{S}x \cap W = (0)$. Then for each vector $y \in V$, there is a scalar $\lambda \in \mathbf{F}$ so that $y + \lambda x$ separates $\mathcal{S}$ and $\mathcal{S}(y + \lambda x) \cap W = (0)$.*

Lemma 1.6. *If $k(\mathcal{S}) = k$, then there exists an M with dim $M = k$ and dim $\mathcal{S}_M^c \leq k$.*

Proof. Since $k(\mathcal{S}) = k$, there exist $x_0 \in H$ and $A_1, ..., A_k \in \mathcal{S}$ such that

$$max\{dim\ [\mathcal{S}x] : x \in H\} = dim\ [A_1x_0, ..., A_kx_0] = k.$$

Let

$$M = [A_1x_0, ..., A_kx_0], \ \widehat{\mathcal{S}} = span\{A_1, ..., A_k\}, \ \text{ and } \ \mathcal{S}_M = \{S \in \mathcal{S} : R(S) \subseteq M\}.$$

It is enough to prove $\mathcal{S} = span\{\widehat{\mathcal{S}} \cup \mathcal{S}_M\}$. Since for any $S \in \mathcal{S}$, there exist $\lambda_1, ..., \lambda_k$ such that $Sx_0 = \sum_{i=1}^{k} \lambda_i A_i x_0$. Let $S_1 = S - \sum_{i=1}^{k} \lambda_i A_i$, then $S_1 x_0 = 0$. If $S_1 = 0$, then $S \in \widehat{\mathcal{S}}$. Next we show that if $S_1 \neq 0$, then $S_1 \in \mathcal{S}_M$.

If $S_1 \notin \mathcal{S}_M$, there exists $y \in H$ such that $S_1 y \notin M = \widehat{\mathcal{S}}x_0$. Let $W = [S_1 y]$. Then $\widehat{\mathcal{S}}x_0 \cap W = (0)$. By Lemma 1.5, there exists $\lambda \in \mathbf{C}$ such that $y + \lambda x_0$ separates $\widehat{\mathcal{S}}$ and $\widehat{\mathcal{S}}(y + \lambda x_0) \cap W = (0)$. Since $S_1 \neq 0$ and $S_1 x_0 = 0$, it follows $\{A_1, ..., A_k, S_1\}$ is linearly independent. Let $\tilde{\mathcal{S}} = span\{A_1, ..., A_k, S_1\}$. Next we prove that $y + \lambda x_0$ separates $\tilde{\mathcal{S}}$. For any $A \in \widehat{\mathcal{S}}, t \in \mathbf{C}$, if $(A + tS_1)(y + \lambda x_0) = 0$, then $A(y + \lambda x_0) = -tS_1 y$. Since $\widehat{\mathcal{S}}(y + \lambda x_0) \cap W = (0)$, it follows that $t = 0$ and $A(y + \lambda x_0) = 0$. Since $y + \lambda x_0$ is a separating vector of $\widehat{\mathcal{S}}$, we have $A = 0$. Hence $y + \lambda x_0$ separates $\tilde{\mathcal{S}}$, which implies $k(\mathcal{S}) \geq k + 1$, a contradiction. □

Definition 1.7. Suppose $\mathcal{S}$ is a subspace of $B(H)$. We say $\mathcal{S}$ has property A if for any subspace $\mathcal{S}_1$ of $\mathcal{S}$, we have $k(\mathcal{S}_1) \geq \{\sqrt{2dim\ \mathcal{S}_1} - 1/2\}$, where $\{t\}$ denotes the smallest integer that is greater than or equal to t.

We say $\mathcal{S}$ has property B if there exists a nonzero subspace M of H such that $k(\mathcal{S}_M) = dim\ M$.

Remark It is clear that if $\mathcal{S}$ has property A, then so does any subspace of $\mathcal{S}$. If $\mathcal{S}$ has property B, then so does any subspace of $B(H)$ containing $\mathcal{S}$.

For $x, y \in H$, let $x \otimes y$ denote the rank-one operator $u \to (u, y)x$.

Lemma 1.8[49]. *Let $A, B \in B(H)$ and $\mathcal{S} = span\{A, B\}$. Then $k(\mathcal{S}) = 1$ if and only if one of the following holds:*

(1) $\dim \mathcal{S} = 1$,

(2) there exist $x_0, x_1, x_2 \in H$ such that $A = x_0 \otimes x_1, B = x_0 \otimes x_2$.

Lemma 1.9. *Suppose $\dim \mathcal{S} = n \geq 2$. If $k(\mathcal{S}) < \{\sqrt{2n} - 1/2\}$, then $\mathcal{S}$ has property B.*

Proof. If $n = 2$, then $k(\mathcal{S}) = 1$. Lemma 1.8 now implies that $\mathcal{S}$ has property B.

Suppose the statement is true for all $\mathcal{S}$ with $2 \leq \dim \mathcal{S} \leq n-1, n \geq 3$. For any $\mathcal{S}$ with $\dim \mathcal{S} = n$, let $k(\mathcal{S}) = k$. By Lemma 1.6, there exists a subspace M of H such that $\dim M = k$ and $\dim \mathcal{S}_M^c \leq k$.

If $\mathcal{S}_M = \mathcal{S}$, clearly $k(\mathcal{S}_M) = k(\mathcal{S}) = \dim M$.

If $\mathcal{S}_M \subset \mathcal{S}$, then let P be the orthogonal projection of H onto M. We have, for any $\mathcal{S}_M^c, P^\perp \mathcal{S}_M^c \neq (0)$, so $k(P^\perp \mathcal{S}_M^c) \geq 1$. Hence $k(\mathcal{S}_M) \leq k-1$, by Proposition 1. 3. Since $k < \{\sqrt{2n} - 1/2\}$, we have $\{\sqrt{2n} - 1/2\} - 1 \leq \{\sqrt{2(n-k)} - 1/2\}$. So $k - 1 < \{\sqrt{2n} - 1/2\} - 1 \leq \{\sqrt{2(n-k)} - 1/2\}$. Hence $k(\mathcal{S}_M) < \{\sqrt{2(n-k)} - 1/2\} \leq \{\sqrt{2 \dim \mathcal{S}_M} - 1/2\}$. (Since $\dim \mathcal{S}_M + \dim \mathcal{S}_M^c = n$, it follows that $\dim \mathcal{S}_M = n - \dim \mathcal{S}_M^c$. Since $\dim \mathcal{S}_M^c \leq k$, it follows that $\dim \mathcal{S}_M \geq n-k$.) By the induction hypothesis, $\mathcal{S}_M$ has property B. It follows that $\mathcal{S}$ has property B. □

Lemma 1.10. *If $\dim \mathcal{S} = n$ and $\mathcal{S}$ has property A then $\mathcal{S}$ is $[\sqrt{2n}\,]$-reflexive, where $[t]$ denotes the largest integer that is less than or equal to t.*

Proof. If $n = 1$, Lemma 1.10[60] implies that $\mathcal{S}$ is reflexive.

Suppose the statement is true for all $\mathcal{S}$ with property A and $\dim \mathcal{S} \leq n-1, n \geq 2$. Suppose $\dim \mathcal{S} = n$, $\mathcal{S}$ has property A, and $k(\mathcal{S}) = k$. Since $\mathcal{S}$ has property $A, k \geq \{\sqrt{2n} - 1/2\}$. If $k = n$, then $\mathcal{S}$ has a separating vector, so $\mathcal{S}$ is 2-reflexive. Hence $\mathcal{S}$ is

$[\sqrt{2n}\,]$-reflexive, since $n \geq 2$ and $[\sqrt{2n}\,] \geq 2$.

Suppose that $\{\sqrt{2n} - 1/2\} \leq k \leq n-1$. Let $m = [\sqrt{2n}]$. Since $k(\mathcal{S}) = k$, there exist $x_1 \in H$ and $\{A_1, ..., A_k\} \subseteq \mathcal{S}$ such that $\{A_i x_1\}_{i=1}^{k}$ is a basis of $\mathcal{S}x_1$. Suppose $\mathcal{S} = span\{A_1, ..., A_n\}$. There exists a unique $k \times n$ complex matrix (a_{ij}) with $a_{ij} = 0 (i \neq j)$, $a_{jj} = 1 (j \leq k)$ and $A_j x_1 = \sum_{i=1}^{k} a_{ij} A_i x_1$, $j = 1, ..., n$. Nest we prove that if $T^{(m)} \in ref(\mathcal{S}^{(m)})$, then $T \in \mathcal{S}$. For any $x_2, ..., x_m \in H$, there exist scalars $t_1, ..., t_n$ such that

$$\begin{pmatrix} Tx_1 \\ \vdots \\ Tx_m \end{pmatrix} = t_1 \begin{pmatrix} A_1 x_1 \\ \vdots \\ A_1 x_m \end{pmatrix} + \ldots + t_n \begin{pmatrix} A_n x_1 \\ \vdots \\ A_n x_m \end{pmatrix}. \tag{1.6}$$

Since $Tx_1 \in span\{A_1 x_1, ..., A_n x_1\}$, there exist $\mu_1, ..., \mu_k$ such that

$$Tx_1 = \sum_{i=1}^{k} \mu_i A_i x_1. \tag{1.7}$$

By (1.6) and (1.7), we have

$$Tx_g = \sum_{i=1}^{k} \mu_i A_i x_g + \sum_{j=1}^{n} t_j (A_j - \sum_{i=1}^{k} a_{ij} A_i) x_g, \; g = 2, ..., m. \tag{1.8}$$

Let

$$T_1 = T - \sum_{i=1}^{k} \mu_i A_i, \text{ and } B_j = A_j - \sum_{i=1}^{k} a_{ij} A_i. \tag{1.9}$$

Note $B_j = 0$ for $j = 1, ..., k$. By (1.8) and (1.9), we have

$$\begin{pmatrix} T_1 x_2 \\ \vdots \\ T_1 x_m \end{pmatrix} = t_{k+1} \begin{pmatrix} B_{k+1} x_2 \\ \vdots \\ B_{k+1} x_m \end{pmatrix} + \ldots + t_n \begin{pmatrix} B_n x_1 \\ \vdots \\ B_n x_m \end{pmatrix}.$$

By the induction hypothesis, we have that $span\{B_{k+1}, ..., B_n\}$ is $[\sqrt{2(n-k)}\,]$-reflexive. Since $k \geq \{\sqrt{2n} - 1/2\}$, we have $[\sqrt{2n}\,] - 1 = m - 1 \geq [\sqrt{2(n-k)}\,]$. It follows that $T_1 \in span\{B_{k+1}, ..., B_n\}$. Therefore $T \in \mathcal{S}$. □

Proposition 1.11. *If $dim\ [\mathcal{S}H] = k$, then $\mathcal{S}$ is k-reflexive.*

Proof. Since $dim\ \mathcal{S} = n$, $\mathcal{S} \subseteq F(H)$, and $dim\ [\mathcal{S}H] = k$, there exists an orthogonal projection P satisfying $dim\ PH = m < \infty$ and $P\mathcal{S}P = \mathcal{S}$. So we may assume that $\mathcal{S}$ is a subspace of $M_m(\mathbf{C})$. Let $\{e_1, ..., e_k\}$ be an orthonormal basis of $\mathcal{S}\mathbf{C}^m \subseteq \mathbf{C}^m$. Extend this to an orthonormal basis $\{e_1, ..., e_k, e_{k+1}, ..., e_m\}$ of $\mathbf{C}^m$. Clearly $\mathcal{S}$ is a subspace of $\mathcal{R} = \{(r_{ij}) \in M_m(\mathbf{C}) : r_{ij} = 0, \text{ for any } i > k\}$. It is easy to prove that $\mathcal{R}^*$ is reflexive. Since $\mathcal{R}^{*(k)}$ has a separating vector, it follows that $\mathcal{R}^{*(k)}$ is elementary, by Proposition 3.2 [8]. By Proposition 2.10 [8], it follows that $\mathcal{S}^{*(k)}$ is reflexive. Hence $\mathcal{S}^{(k)}$ is reflexive. □

Theorem 1.12. *If $dim\ \mathcal{S} = n, k(\mathcal{S}) = k$, then $\mathcal{S}$ is k-reflexive.*

Proof. If $\mathcal{S}$ has property A, by Lemma 1.10, we have $\mathcal{S}$ is $[\sqrt{2n}\ ]$-reflexive. Since $k \geq \{\sqrt{2n} - 1/2\} \geq [\sqrt{2n}\ ]$, it follows that $\mathcal{S}$ is k-reflexive.

(i) Suppose $\mathcal{S}$ does not have property A. Thus there exists a subspace $\mathcal{S}_1$ of $\mathcal{S}$ such that $k(\mathcal{S}_1) < \{\sqrt{2n} - 1/2\}$. By Lemma 1.9, $\mathcal{S}_1$ has property B. Hence $\mathcal{S}$ has property B.

(ii) Let M be a maximal subspace of H such that $k(\mathcal{S}_M) = dim\ M$. Let P be the orthogonal projection of H onto M.

If $\mathcal{S}_M \subset \mathcal{S}$, we prove next $P^\perp\mathcal{S}$ has property A. If property A fails, then (i) implies that $P^\perp\mathcal{S}$ has property B. Thus there exists a subspace N of H such that

$$k((P^\perp\mathcal{S})_N) = dim\ N. \tag{1.10}$$

By (1.10), we have $N \subseteq P^\perp H$. Let $\tilde{M} = M \oplus N$. By Proposition 1.3,

$$\begin{aligned} k(\mathcal{S}_{\tilde{M}}) &\geq k((\mathcal{S}_{\tilde{M}})_M) + k(P^\perp(\mathcal{S}_{\tilde{M}})^c_M) \\ &= k(\mathcal{S}_M) + k(P^\perp\mathcal{S}_{\tilde{M}}) \end{aligned}$$

$$
\begin{aligned}
&= k(P^{\perp}\mathcal{S}_{\tilde{M}}) + dim\ M \\
&= k((P^{\perp}\mathcal{S})_{\tilde{M}}) + dim\ M \\
&= k((P^{\perp}\mathcal{S})_{N}) + dim\ M \\
&= dim\ N + dim\ M = dim\ \tilde{M}.
\end{aligned}
$$

Thus $k(\mathcal{S}_{\tilde{M}}) = dim\ \tilde{M}$, contradicting the maximality of M.

Suppose $dim\ M = m$ and $dim\ (P^{\perp}\mathcal{S}) = l$. Let $r = [\sqrt{2l}]$. We show $\mathcal{S}$ is $(m+r)$-reflexive by induction on l.

If $l = 0$, then $[\mathcal{S}H] = M$. By Proposition 1.11, it follows that $\mathcal{S}$ is m-reflexive.

Suppose the statement is true for all $dim\ (P^{\perp}\mathcal{S}) \le l-1$, $l \ge 1$. Suppose $dim\ P^{\perp}\mathcal{S} = l$. Since $\mathcal{S} = \mathcal{S}_M + \mathcal{S}_M^c$, we have $P^{\perp}\mathcal{S} = P^{\perp}\mathcal{S}_M^c$. If $\{A_1, ..., A_s\}$ is a basis of $\mathcal{S}_M^c$, we can easily prove that $\{P^{\perp}A_i\}_{i=1}^s$ is linearly independent, so $s = l$. If $k(P^{\perp}\mathcal{S}) = J$, then there exists an $x_1 \in H$ and $\{A_1, ..., A_J\} \subseteq \mathcal{S}_M^c$ so that $\{P^{\perp}A_1x_1, ..., P^{\perp}A_Jx_1\}$ is linearly independent. Let $\{A_{l+1}, ..., A_n\}$ be a basis of $\mathcal{S}_M$. It follows that $\{A_1, ..., A_n\}$ is a basis of $\mathcal{S}$. Since $P^{\perp}A_jx_1 \in span\{P^{\perp}A_1x_1, ..., P^{\perp}A_Jx_1\}, J+1 \le j \le n$, we have

$$
P^{\perp}A_jx_1 = \sum_{i=1}^{J} a_{ij}P^{\perp}A_ix_1, J+1 \le j \le l \text{ and } P^{\perp}A_jx_1 = 0, l+1 \le j \le n. \tag{1.11}
$$

If $T \in B(H)$ and $T^{(m+r)} \in ref(\mathcal{S}^{(m+r)})$. For any $x_2, ..., x_{m+r} \in H$, there exist $t_1, ..., t_n$ so that

$$
\begin{pmatrix} Tx_1 \\ \vdots \\ Tx_{m+r} \end{pmatrix} = t_1 \begin{pmatrix} A_1x_1 \\ \vdots \\ A_1x_{m+r} \end{pmatrix} + ... + t_n \begin{pmatrix} A_nx_1 \\ \vdots \\ A_nx_{m+r} \end{pmatrix}. \tag{1.12}
$$

Since $Tx_1 \in span\{A_1x_1, ..., A_nx_1\}$, it follows that $P^{\perp}Tx_1 \in span\{P^{\perp}A_1x_1, ..., P^{\perp}A_Jx_1\}$.

Hence there exist $v_1, ..., v_J$ so that

$$P^{\perp}Tx_1 = \sum_{i=1}^{J} v_i P^{\perp} A_i x_1. \tag{1.13}$$

By (1.11) to (1.13), we have

$$Tx_g = \sum_{i=1}^{J}(v_i - \sum_{j=J+1}^{l} t_j a_{ij}) A_i x_g + \sum_{i=J+1}^{n} t_i A_i x_g,\ g = 2, ..., m+r. \tag{1.14}$$

Let

$$C = T - \sum_{i=1}^{J} v_i A_i,\ B_j = A_j - \sum_{i=1}^{J} a_{ij} A_i,\ J+1 \le j \le l,\ B_j = A_j,\ l+1 \le j \le n. \tag{1.15}$$

By (1.14) and (1.15), we have

$$\begin{pmatrix} Cx_2 \\ \vdots \\ Cx_{m+r} \end{pmatrix} = t_{J+1} \begin{pmatrix} B_{J+1}x_2 \\ \vdots \\ B_{J+1}x_{m+r} \end{pmatrix} + ... + t_n \begin{pmatrix} B_n x_2 \\ \vdots \\ B_n x_{m+r} \end{pmatrix}.$$

Let $\tilde{\mathcal{S}} = span\{B_{J+1}, ..., B_n\}$. Then $dim\ P^{\perp}\tilde{\mathcal{S}} \le l - J$ and $k(\tilde{\mathcal{S}}_M) = k(\mathcal{S}_M) = dim\ M$. Since $P^{\perp}\mathcal{S}$ has property A, we have that $J \ge \{\sqrt{2l} - 1/2\}$. So $m + r - 1 \ge m + [\sqrt{2(l-J)}\ ] \ge m + [\sqrt{2dim\ P^{\perp}\tilde{\mathcal{S}}}\ ]$. By the induction hypothesis, we have $C \in span\{B_{J+1}, ..., B_n\}$. Hence $T \in span\{A_1, ..., A_n\} = \mathcal{S}$. By Proposition 1.4, $k = k(\mathcal{S}_M) + k(P^{\perp}\mathcal{S}_M^c) = m + k(P^{\perp}\mathcal{S})$. Since $P^{\perp}\mathcal{S}$ has property A, and $k(P^{\perp}\mathcal{S}) \ge \{\sqrt{2l} - 1/2\}$, it follows that $k \ge m + \{\sqrt{2l} - 1/2\} \ge m + [\sqrt{2l}\ ]$. Hence $\mathcal{S}$ is k-reflexive.

If $\mathcal{S}_M = \mathcal{S}$, then $\mathcal{S}$ is k-reflexive by Proposition 1.11. □

Theorem 1.13. *If $dim\ \mathcal{S} = n$, then $\mathcal{S}$ is $[\sqrt{2n}\ ]$-reflexive.*

Proof. If $n = 1$, 2 and 3, Theorem 3 [70] implies the result. Suppose the result holds for $dim\ \mathcal{S} \le n - 1, n \ge 4$. Let $dim\ \mathcal{S} = n$ and suppose $k(\mathcal{S}) = k$. If $k \le [\sqrt{2n}\ ]$, by Proposition 1.12, it follows that $\mathcal{S}$ is $[\sqrt{2n}\ ]$-reflexive.

If $k > [\sqrt{2n}\ ]$ then $k \geq \{\sqrt{2n} - 1/2\}$. If $k = n$, then $\mathcal{S}$ is 2-reflexive. Hence $\mathcal{S}$ is $[\sqrt{2n}\ ]$-reflexive. If $[\sqrt{2n}\ ] < k \leq n-1$, using the same argument as Lemma 1.10, we have $dim\ span\{B_{k+1}, ..., B_n\} \leq n-k$. By the induction hypothesis, it follows that $span\{B_{k+1}, ..., B_n\}$ is $[\sqrt{2(n-k)}\ ]$-reflexive. Since $k \geq \{\sqrt{2n}-1/2\}$, it follows that $[\sqrt{2n}\ ]-1 \geq [\sqrt{2(n-k)}\ ]$. Thus $span\{B_{k+1}, ..., B_n\}$ is $([\sqrt{2n}\ ]-1)$-reflexive, so $\mathcal{S}$ is $[\sqrt{2n}]$-reflexive.
□

Example 1.14. Let $\mathcal{S}_k$ be the set of all $k \times k$ upper triangular matrices with zero trace. We may show $dim\ \mathcal{S}_k = \frac{k(k+1)}{2} - 1$ and $\mathcal{S}_k$ is not $(k-1)$-reflexive. For any positive integer l, one can easily show that there exists a positive integer k such that

$$\frac{k(k+1)}{2} - 1 \leq l < \frac{(k+1)(k+2)}{2} - 1. \tag{1.16}$$

For any positive integer l, choose k such that (1.16) holds and let $m = l - (\frac{k(k+1)}{2} - 1)$. Let $\mathcal{S} = \mathcal{S}_k \oplus \mathcal{A}_m$, where $\mathcal{A}_m = \{diag(a_1, ..., a_m) : a_i \in \mathbf{C}\}$. It is easy to prove that $\mathcal{S}$ is not $([\sqrt{2l}] - 1)$-reflexive.

Remarks (1) Theorem 1.13 answers a question of Magajna [85]. It indicates that if $\mathcal{S}$ is n-dimensional subspace of $B(H)$, then $[\sqrt{2n}\ ]$ is the smallest integer such that all n-dimensional subspaces of $B(H)$ are $[\sqrt{2n}\ ]$-reflexive.

(2) By the proof of Theorem 1.13, we have that if $k(\mathcal{S}) \geq n-1$, then $\mathcal{S}$ is 2-reflexive and that if $k(\mathcal{S}) \geq n-4$, then $\mathcal{S}$ is 3-reflexive. This improves Theorem 3.6 [23].

In the following, we give an application of Theorem 1.13.

Theorem 1.15. *If* $\Phi(\cdot) = \sum_{i=1}^{n} a_i(\cdot)b_i$ *is an elementary operator on* $\mathcal{A}$, $\{a_i\}, \{b_i\}$ *are subsets of a* C^**-algebra* $\mathcal{A}$, *then* Φ *is completely positive if and only if* Φ *is* $max\{[\sqrt{2(n-1)}\ ], 1\}$*-positive.*

The proof is similar to the proof of Theorem 6 [13], so we leave it to the reader.

Remark In section 2.2.1, we improve Theorem 1.15. We show that if ϕ is $[\sqrt{n}]$-positive, then ϕ is completely positive.

1.2 Decomposability of finite rank operators

Let H be a complex Hilbert space. By a *subspace lattice* on H, we mean a collection $\mathcal{L}$ of subspaces of H with (0), H in $\mathcal{L}$ and such that for every family $\{M_r\}$ of elements of $\mathcal{L}$, both $\cap M_r$ and $\vee M_r$ belong to $\mathcal{L}$, where $\vee M_r$ denotes the closed linear span of $\{M_r\}$. Let $\mathcal{L}$ be a subspace lattice on H and let $\mathcal{L}^\perp = \{I - P : P \in \mathcal{L}\}$. We have $alg\mathcal{L}^\perp = (alg\mathcal{L})^*$. A totally ordered subspace lattice is called a *nest*. A subspace lattice $\mathcal{L}$ is *distributive* if $K \cap (L \vee M) = (K \cap L) \vee (K \cap M)$ holds identically in $\mathcal{L}$. We say that $\mathcal{L}$ is *complemented* if for every $L \in \mathcal{L}$, there exists $L' \in \mathcal{L}$ such that $L' \cap L = (0)$ and $L' \cap L = H$. A complemented and distributive subspace lattice is called a *Boolean lattice*. An element L of a subspace lattice $\mathcal{L}$ is called an *atom* if the condition $(0) \subseteq K \subseteq L$ $(K \in \mathcal{L})$ implies either $K = (0)$ or $K = L$. A subspace lattice is *atomic* if each element of the lattice is the closed linear span of the atoms it contains. If K, $L \in \mathcal{L}$, we denote by L_- the subspace $L_- = \vee\{M \in \mathcal{L} : L \not\subseteq M\}$, by $K_\# = \vee\{L \in \mathcal{L} : K \not\subseteq L_-\}$ and by $K_+ = \wedge\{L \in \mathcal{L} : L \not\subseteq K\}$. By convention $H_+ = \cap\emptyset = H$, $(0)_- = \vee\emptyset = (0)$. Complete distributivity is a much stronger condition than distributivity. The complete distributivity of $\mathcal{L}$ is equivalent to $K = K_\#$ for all $K \in \mathcal{L}$. An element L in $\mathcal{L}$ is *completely meet prime* if $L \not\supseteq L_+$. An element M in $\mathcal{L}$ is *completely joint prime* if $M \not\subseteq M_-$.

If M is a subset of H, we denote by $[M]$ the norm closure of $span\{x : x \in M\}$. Let R and T be finite rank operators on H. We say that R is a *summand* of T if $rankT = rankR + rank(T - R)$. If S is a subset of $B(H)$, S is said to be *rank decomposable* if each finite rank operator in S is a sum of rank one operators in S. We say that S is *strongly rank decomposable* if, for each $r > 1$, each operator in $\mathcal{S}$ of rank r can be expressed as the sum of r rank one operators in $\mathcal{S}$.

Finite rank operators and rank one operators have been used extensively in the study of nest algebras and related non-self-adjoint reflexive algebras. By [46], we know that if $\mathcal{L}$ is a nest or an atomic Boolean subspace lattice on H, then $alg\mathcal{L}$ is strongly rank decomposable. In [75], we improve these results. Erdos and Power [30] prove that if $\mathcal{N}$ is a nest and $\mathcal{S}$ is a σ-weakly closed bimodule of $alg\mathcal{N}$, then $\mathcal{S}$ is strongly rank decomposable. In [48], Hopenwasser and Moore construct a totally atomic commutative subspace lattice $\mathcal{L}$ and a rank two operator in $alg\mathcal{L}$ which cannot be written as a sum of rank one operators in $alg\mathcal{L}$.

Let $\mathcal{S}$ be either a reflexive subspace or a bimodule of a reflexive algebra. In this section, we find some conditions which imply T has a rank one summand in $\mathcal{S}$, where $T \in \mathcal{S} \cap F(H)$. We also obtain some necessary and sufficient conditions such that $\mathcal{S}$ is strongly rank decomposable. For $n \geq 3$, we construct an atomic Boolean subspace lattice $\mathcal{L}$ on H with n atoms for which there is a finite rank operator T in $\mathcal{S}(\mathcal{L})$ such that T does not have a rank one summand in $\mathcal{S}(\mathcal{L})$, where $\mathcal{S}(\mathcal{L})$ is the set of all operators on H that annihilate all the operators of rank at most one in $alg\mathcal{L}$. This answers a question in [56] negatively. We obtain some lattice-theoretic conditions on a subspace lattice $\mathcal{L}$ which imply $alg\mathcal{L}$ is strongly rank decomposable. Theorems 2.12 and 2.13 generalize the main results of [75].

In [29], Erdos gives some necessary and sufficient conditions such that a reflexive subspace of $B(H)$ contains a rank one operator. In the following we obtain another equivalent condition.

Lemma 2.1. *Let $\mathcal{S}$ be a reflexive subspace of $B(H)$. Then $e \otimes f$ belongs to $\mathcal{S}$ if and* *only if* $f \in (span\{y : e \notin [\mathcal{S}y],\ y \in H\})^{\perp}$.

Proof. Suppose that $e \otimes f \in \mathcal{S}$. Since $\mathcal{S}$ is reflexive, it follows that for any y in H, $e \otimes f(y) = (y, f)e \in [\mathcal{S}y]$. Hence if $e \notin [\mathcal{S}y]$, $(y, f) = 0$. So $f \in (span\{y : e \notin [\mathcal{S}y],\ y \in H\})^{\perp}$.

Conversely, suppose $f \in (span\{y : e \notin [\mathcal{S}y],\ y \in H\})^{\perp}$. Let $y \in H$. Since $e \otimes f(y) = (y, f)e$ and $f \in (span\{y : e \notin [\mathcal{S}y],\ y \in H\})^{\perp}$, it follows that $e \otimes f(y) = (y, f)e \in [\mathcal{S}y]$. Since $\mathcal{S}$ is reflexive, it follows that $e \otimes f \in \mathcal{S}$. □

The following Lemma will be used repeatedly.

Lemma 2.2[46]. *Let T be a finite rank operator and let A be a rank one operator in $B(H)$. Then A is a summand of T if and only if A is of the form $(Ty) \otimes (T^*f)$ (or equivalently, $T(y \otimes f)T$), where y and f are vectors in H and $(Ty, f) = 1$.*

Theorem 2.3. *Suppose that $\mathcal{S}$ is a reflexive subspace of $B(H)$ and T is a finite rank operator in $\mathcal{S}$. Then T has a rank one summand in $\mathcal{S}$ if and only if there is an e in H such that $e \in T(H)$ and $e \notin span\{Ty : e \notin [\mathcal{S}y],\ y \in H\}$, where $T(H)$ is the range of T.*

Proof. Suppose that $e \in T(H)$ and $e \notin span\{Ty : e \notin [\mathcal{S}y],\ y \in H\}$. Choose $g \in H$ such that $g \in (span\{Ty : e \notin [\mathcal{S}y],\ y \in H\})^{\perp}$, $(e, g) = 1$, and take $y \in H$ such that $Ty = e$. Thus $(Ty, g) = (y, T^*g) = 1$. Hence for any $\tilde{y}$ satisfying $e \notin [\mathcal{S}\tilde{y}]$, we have $(\tilde{y}, T^*g) = 0$. It follows that $T^*g \in (span\{y : e \notin [\mathcal{S}y],\ y \in H\})^{\perp}$. Using Lemma 2.1, $e \otimes T^*g = (Ty) \otimes (T^*g) \in \mathcal{S}$. Using Lemma 2.2, $e \otimes (T^*g) = (Ty) \otimes (T^*g)$ is a rank one summand of T in $\mathcal{S}$.

Conversely, suppose that T has a rank one summand in $\mathcal{S}$. By Lemma 2.2, there exist

m and f in H such that

$$T(m \otimes f)T = (Tm) \otimes (T^* f) \in \mathcal{S},$$

and

$$(Tm, f) = 1 = (m, T^* f).$$

Let $Tm = e$. Using Lemma 2.1, we have $T^* f \in (span\{y : e \notin [\mathcal{S}y],\ y \in H\})^{\perp}$. Hence for any $v \in span\{y : e \notin [\mathcal{S}y],\ y \in H\}$, $(v, T^* f) = (Tv, f) = 0$. Since $(e, f) = (Tm, f) = 1$, it follows that $e \notin T(span\{y : e \notin [\mathcal{S}y],\ y \in H\}) = span\{Ty : e \notin [\mathcal{S}y],\ y \in H\}$. □

Corollary 2.4. *Let M and N be nonzero subspaces of H satisfying $M \cap N = 0$ and $M \vee N = H$ and let $\mathcal{L} = \{(0), M, N, H\}$. Then every σ-weakly closed alg$\mathcal{L}$-bimodule $\mathcal{S}$ is strongly rank decomposable.*

Proof. By Theorem 2.2[61] and Theorem 3.1[4], it follows that $\mathcal{S}$ is reflexive. By Theorem 2[45], we know that $\mathcal{S}$ is determined by an order homomorphism ϕ of $\mathcal{S}$. Let ϕ be any order homomorphism of $\mathcal{L}$ and let

$$\mathcal{M} = \{T \in B(H) : (I - \phi(E))TE = 0, E \in \mathcal{L}\}.$$

By the symmetry of M and N, we only need to prove $\mathcal{M}$ has strong rank decomposability in the following cases.

(1) $\phi : M \mapsto M,\ N \mapsto 0,$

(2) $\phi : M \mapsto N,\ N \mapsto 0,$

(3) $\phi : M \mapsto N,\ N \mapsto M,$

(4) $\phi : M \mapsto M,\ N \mapsto N,$

(5) $\phi : M \mapsto H,\ N \mapsto 0,$

For cases (1) to (4), we can easily prove the result using Theorem 2.3.

In case (5), $\mathcal{M} = \{T \in B(H) : TN = 0\}$. Let P denote the projection on N. Then T is in $\mathcal{M}$ if and only if $TP = 0$. Hence $\mathcal{M}$ has strong rank decomposability. □

Remark In Corollary 2.4, we can prove that if $\mathcal{V}$ is any normed closed subspace of $B(H)$ which is $alg\mathcal{L}$-bimodule, then $\mathcal{V}$ is strongly rank decomposable.

If $\mathcal{L}$ is a subspace lattice on the Hilbert space H, let $\mathcal{S}(\mathcal{L})$ denote the set of all operators on H that annihilate all the operators of rank at most one in $alg\mathcal{L}$, that is

$$\mathcal{S}(\mathcal{L}) = \{T \in B(H) :\ tr(TR) = 0, \text{ for every } R \in alg\mathcal{L} \text{ of rank at most one}\}.$$

Thus $\mathcal{S}(\mathcal{L})$ is an $alg\mathcal{L}$-bimodule.

Lemma 2.5[56]. *For any subspace lattice $\mathcal{L}$ on H,*

$$\mathcal{S}(\mathcal{L}) = \{T \in B(H) :\ T(K) \subseteq K_- \text{ for every } K \in \mathcal{L}\}.$$

Lemma 2.6[56]. *Let $\mathcal{L}$ be a subspace lattice on H and $e, f \in H$. The following are equivalent.*

(1) $e \otimes f \in \mathcal{S}(\mathcal{L})$,

(2) $e \in L$ *and* $f \in (L_{\#})^{\perp}$ *for some* $L \in \mathcal{L}$.

Theorem 2.7. *Let $\mathcal{L}$ be a subspace lattice and let $T \in \mathcal{S}(\mathcal{L}) \cap F(H)$. Then T has a rank one summand in $\mathcal{S}(\mathcal{L})$ if and only if there exists an $L \in \mathcal{L}$ such that $T(H) \cap L \not\subseteq T(L_{\#})$.*

Proof. Suppose that there exists $L \in \mathcal{L}$ such that $T(H) \cap L \not\subseteq T(L_{\#})$. Choose g in H such that $v \in L_{\#}$, $(Tv, g) = 0$, and let $e \in L$ such that $Ty = e$, $(e, g) = (Te, g) = 1$. We have

$$(Ty, g) = (y, T^{*}g) = 1 \text{ and } T(y \otimes g)T = (Ty) \otimes (T^{*}g) \in \mathcal{S}(\mathcal{L}).$$

By Lemma 2.2, it follows that T has a rank one summand in $\mathcal{S}(\mathcal{L})$.

Conversely, suppose T has a rank one summand in $\mathcal{S}(\mathcal{L})$. By Lemma 2.2, there exist e, f in H such that

$$T(e \otimes f)T = (Te) \otimes (T^* f) \in \mathcal{S}(\mathcal{L}) \text{ and } (Te, f) = 1.$$

By Lemma 2.6, there exists L in $\mathcal{L}$ such that $Te \in L$ and $T^* f \in (L_\#)^\perp$. Since $Te \in L$, $(Te, f) = 1$ and for any $v \in L_\#$, $(Tv, f) = 0$, we have that $T(H) \cap L \not\subseteq T(L_\#)$. □

Example 2.8. For $n \geq 3$, there is an atomic Boolean subspace lattice $\hat{\mathcal{L}}$ with three atoms such that $\mathcal{S}(\hat{\mathcal{L}})$ is not strongly decomposable.

Proof. Let H be a finite dimensional Hilbert space and let A be an invertible operator in $B(H)$. Define $L_1 = \{(x, 0, 0) : x \in H\}$, $L_2 = \{(x, Ax, 0) : x \in H\}$ and $L_3 = \{(x, Ax, Ax) : x \in H\}$. By Lemma 6.3[2], it follows that $\{L_1, L_2, L_3\}$ is the set of atoms of an atomic Boolean subspace lattice.

Define $T : L_1 \to L_2 \vee L_3$, by $(x, 0, 0) \mapsto (0, 0, Px)$, $T : L_2 \to L_1 \vee L_3$, by $(x, Ax, 0) \mapsto (0, Px, Px)$, and $T : L_3 \to L_2 \vee L_1$, by $(x, Ax, Ax) \mapsto (0, Px, 0)$, where P is a finite projection in $B(H)$. We can extend T to a bounded finite rank operator in $B(H \oplus H \oplus H)$. By the definition of T, it follows that $T \in \mathcal{S}(\mathcal{L})$. We have that $T(H) \cap L_1 = 0$, $T(H) \cap L_2 = 0$ and $T(H) \cap L_3 = 0$. We can check that $T(H) \cap (L_2 \vee L_3) \subset T(L_2 \vee L_3)$, $T(H) \cap (L_2 \vee L_1) \subset T(L_2 \vee L_1)$ and $T(H) \cap (L_1 \vee L_3) \subset T(L_1 \vee L_3)$. Hence by Theorem 2.7, T does not have a rank one summand in $\mathcal{S}(\mathcal{L})$, where $\mathcal{L}$ is the subspace lattice generated by L_1, L_2 and L_3. Let $m = n - 3$ and let $\mathcal{L}_1$ be an atomic Boolean subspace lattice with m atoms on Hilbert space H_1. Define

$$\hat{\mathcal{L}} = \mathcal{L} \times \mathcal{L}_1 = \{L \oplus M |\ L \in \mathcal{L},\ M \in \mathcal{L}_1\}.$$

Then $\hat{\mathcal{L}}$ is an atomic Boolean subspace lattice on $H \oplus H_1$ with n atoms. Since

$$alg\hat{\mathcal{L}} = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} : A \in alg\mathcal{L},\ B \in alg\mathcal{L}_1 \right\},$$

it follows that

$$\mathcal{S}(\hat{\mathcal{L}}) = \left\{ \begin{pmatrix} A_{11} & A_{12} \\ A_{12} & A_{22} \end{pmatrix} : A_{11} \in \mathcal{S}(\mathcal{L}), A_{22} \in \mathcal{S}(\mathcal{L}_1), A_{12} \in B(H_1, H) \text{ and } A_{21} \in B(H, H_1) \right\}.$$

Let

$$\hat{T} = \begin{pmatrix} T & 0 \\ 0 & 0 \end{pmatrix}.$$

Then $\hat{T}$ does not have a rank one summand in $\mathcal{S}(\hat{\mathcal{L}})$.

Remark The above Example answers a question in [56, p31] negatively.

Theorem 2.9. *Suppose that $\mathcal{L}$ is a subspace lattice and $Rad(alg\mathcal{L})$ is the radical of $alg\mathcal{L}$. Let $T \in Rad(alg\mathcal{L}) \cap F(H)$. Then T has a rank one summand in $Rad(alg\mathcal{L})$ if and only if there exists an M in $\mathcal{L}$ such that $T(H) \cap M \not\subseteq T(M_- \vee M)$.*

Proof. Suppose that $T(H) \cap M \not\subseteq T(M_- \vee M)$. Choose g in $(T(M_- \vee M))^\perp$, e in H such that $(Te, g) = 1$ and $Te \in M$. Then $(e, T^*g) = 1$, $(Tx, g) = (x, T^*g) = 0$ for any $x \in M_- \vee M$. By $T^*g \in (M_- \vee M)^\perp$, $Te \in M$ and Lemma 3[55], it follows that $(Te) \otimes (T^*g) \in Rad(alg\mathcal{L})$. By Lemma 2.2, T has a rank one summand in $Rad(alg\mathcal{L})$.

Conversely, suppose T has rank one summand in $Rad(alg\mathcal{L})$. It follows that there exist $e, f \in H$ such that $T(e \otimes f)T = (Te) \otimes (T^*f) \in Rad(alg\mathcal{L})$. By Lemma 3[55], there exists M in $\mathcal{L}$ such that $T^*f \in (M_- \vee M)^\perp$, $(Te, f) = 1$. Hence $T(H) \cap M \not\subseteq T(M_- \vee M)$. □

Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be subspace lattices on Hilbert spaces H_1 and H_2. Then the ordinal sum

$\mathcal{L}_1 + \mathcal{L}_2$ is defined as the set of subspaces of $H_1 \oplus H_2$ given by

$$\mathcal{L}_1 + \mathcal{L}_2 = \{L \oplus 0 : \ L \in \mathcal{L}_1\} \cup \{H_1 \oplus M : \ M \in \mathcal{L}_2\}.$$

Theorem 2.10. *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be subspace lattices on Hilbert spaces H_1 and H_2. If $\mathcal{S}(\mathcal{L}_1)$ and $\mathcal{S}(\mathcal{L}_2)$ are strongly rank decomposable, then $\mathcal{S}(\mathcal{L}_1 + \mathcal{L}_2)$ is strongly rank decomposable.*

Proof. Since

$$alg(\mathcal{L}_1 + \mathcal{L}_2) = \left\{ \begin{pmatrix} A_1 & T \\ 0 & A_2 \end{pmatrix} : \ A_i \in alg\mathcal{L}_i, \ \text{ for } i = 1, 2, \ T \in B(H_2, H_1) \right\}.$$

we have

$$\mathcal{S}(\mathcal{L}_1 + \mathcal{L}_2) = \left\{ \begin{pmatrix} B_1 & A \\ 0 & B_2 \end{pmatrix} : \ B_i \in \mathcal{S}(\mathcal{L}_i), \ \text{ for } i = 1, 2 \text{ and } A \in B(H_2, H_1) \right\}. \qquad (2.1)$$

Let T be a finite rank operator in $\mathcal{S}(\mathcal{L}_1 + \mathcal{L}_2)$. Then

$$T = \begin{pmatrix} T_1 & A \\ 0 & T_2 \end{pmatrix}, \quad \text{where } T_i \in \mathcal{S}(\mathcal{L}_i) \text{ for } i = 1, \ 2 \text{ and } A \in B(H_2, H_1).$$

Suppose $T_1 \neq 0$. Since $\mathcal{S}(\mathcal{L}_1)$ is strongly rank decomposable, we may choose $e_1 \in H_1$, $f_1 \in H_1$ such that $T_1(e_1 \otimes f_1)T_1$ is a rank one summand of T_1 in $\mathcal{S}(\mathcal{L}_1)$. Let $e = e_1 \oplus 0$. For any $x = x_1 \oplus x_2 \in H_1 \oplus H_2$, let $f = f_1 \oplus 0 \in H_1 \oplus H_2$, then $(x, f) = (x_1, f_1)$. It follows that

$$T(e \otimes f)T = \begin{pmatrix} T_1(e_1 \otimes f_1)T_1 & T_1(e_1 \otimes f_1)TA \\ 0 & 0 \end{pmatrix}.$$

Since $(Te, f) = (T_1 e_1, f_1) = 1$, (2.1) and $T(e \otimes f)T \in (\mathcal{L}_1 + \mathcal{L}_2)$, it follows from Lemma 2.2 that $T(e \otimes f)T$ is a rank one summand of T in $\mathcal{S}(\mathcal{L}_1 + \mathcal{L}_2)$.

If $T_1 = 0$ and $T_2 \neq 0$, we can similarly prove that T has a rank one summand in $\mathcal{S}(\mathcal{L}_1 + \mathcal{L}_2)$.

Suppose that $T_1 = T_2 = 0$. Then $T = \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix}$. Since $B(H_2, H_1)$ is strongly rank decomposable, it follows that T has a rank one summand in $\mathcal{S}(\mathcal{L}_1 + \mathcal{L}_2)$.

Since T is any finite rank operator in $\mathcal{S}(\mathcal{L}_1 + \mathcal{L}_2)$, it follows that $\mathcal{S}(\mathcal{L}_1 + \mathcal{L}_2)$ is strongly rank decomposable. □

Let

$$\mathcal{J}_{\mathcal{L}} = \{L \in \mathcal{L} : \ L \neq (0) \text{ and } L_- \neq H\}, \ \mathcal{P}_{\mathcal{L}} = \{L \in \mathcal{L} : L \not\subseteq L_-\}.$$

By [17], we know that $L \in \mathcal{L}$ is completely meet prime if and only if $L = M_-$ for some $M \in \mathcal{P}_{\mathcal{L}}$.

Lemma 2.11[90]. *Let K and L be subspaces of H and let $F = \sum_{i=1}^{n} e_i \otimes f_i$ be a rank n operator in $B(H)$. If $F(L) \subseteq K$ and $f_1 \notin L^{\perp}$, then F can be written as $F = \tilde{e}_1 \otimes f_1 + \sum_{i=2}^{n} e_i \otimes \tilde{f}_i$ with $\tilde{e}_1 \in K$.*

Theorem 2.12. *Let $\mathcal{L}$ be a subspace lattice on H such that $\mathcal{J}_{\mathcal{L}} = \mathcal{P}_{\mathcal{L}}$ and $\vee\{L : L \in \mathcal{J}_{\mathcal{L}}\} = H$. Then $alg\mathcal{L}$ is strongly rank decomposable.*

Proof. Suppose that $alg\mathcal{L}$ is not strongly rank decomposable. Then there is a rank n operator $T = \sum_{i=1}^{n} e_i \otimes f_i$ in $alg\mathcal{L}$ such that T does not have a rank one summand in $alg\mathcal{L}$. Since $H = \vee\{M : M \in \mathcal{J}_{\mathcal{L}}\}$, it follows that there exists an M in $\mathcal{J}_{\mathcal{L}}$ such that $f_1 \notin M^{\perp}$. By Lemma 2.11, T can be written as

$$T = \tilde{e}_1 \otimes f_1 + \sum_{i=2}^{n} e_i \otimes \tilde{f}_i,$$

with $\tilde{e}_1 \in M$. Let

$$N = \cap\{L \in \mathcal{J}_{\mathcal{L}} : \tilde{e}_1 \in L\}. \tag{2.2}$$

Then $N \in \mathcal{J}_{\mathcal{L}}$ and $\tilde{e}_1 \in N$.

Now we show that $\tilde{e}_1 \in N_-$. Suppose $\tilde{e}_1 \notin N_-$. Since $F^* = f_1 \otimes \tilde{e}_1 + \sum_{i=2}^{n} \tilde{f}_i \otimes e_i$, by Lemma 2.11, we have that $F^* = g_1 \otimes \tilde{e}_1 + \sum_{i=2}^{n} \tilde{f}_i \otimes h_i$ with $g_1 \in (N_-)^\perp$.

By $\tilde{e}_1 \in N$ and $g_1 \in (N_-)^\perp$, we have that $g_1 \otimes \tilde{e}_1$ is a rank one summand of F^* in $alg\mathcal{L}^\perp$. Hence F has a rank one summand in $alg\mathcal{L}$, a contradiction.

Let $W = N_- \cap N$. We have $\tilde{e}_1 \in W$ and $W \in \mathcal{J}_\mathcal{L}$. By the assumption, $W \subset N$ and $\tilde{e}_1 \in W$. It contradicts (2.2). □

Theorem 2.13. *Let $\mathcal{L}$ be a subspace lattice on H such that $\mathcal{J}_\mathcal{L} = \mathcal{P}_\mathcal{L}$ and $\cap\{L_- : L \in \mathcal{J}_\mathcal{L}\} = 0$. Then $alg\mathcal{L}$ is strongly rank decomposable.*

Proof. By Proposition 2.1[84], it follows that

$$\mathcal{J}_{\mathcal{L}^\perp} = \{(M_-)^\perp : M \in \mathcal{J}_\mathcal{L}\}.$$

Since $\mathcal{J}_\mathcal{L} = \mathcal{P}_\mathcal{L}$, for any $M \in \mathcal{J}_\mathcal{L}$, we have that $(M_-)^\perp$ is completely joint prime. Hence for subspace lattice $\mathcal{L}^\perp$, we have $\mathcal{J}_{\mathcal{L}^\perp} = \mathcal{P}_{\mathcal{L}^\perp}$. Since $\cap\{M_- : M \in \mathcal{J}_\mathcal{L}\} = 0$, it follows that $\vee\{N : N \in \mathcal{J}_{\mathcal{L}^\perp}\} = H$. Since $alg\mathcal{L}$ is strongly rank decomposable if and only if $alg\mathcal{L}^\perp$ is strongly rank decomposable, by Theorem 2.12, to prove the theorem, it is sufficient to show that $(L_-)^\perp \not\subseteq ((L_-)^\perp)_-$ for any $L \in \mathcal{P}_\mathcal{L}$. Since $((L_-)^\perp)_- = \vee\{M^\perp : M^\perp \not\supseteq (L_-)^\perp, M^\perp \in \mathcal{L}^\perp\}$, it follows that

$$((L_-)^\perp)_- \subseteq L^\perp. \tag{2.3}$$

Suppose $(L_-)^\perp \subseteq ((L_-)^\perp)_-$. By (2.3), it follows that $(L_-)^\perp \subseteq L^\perp$. Hence $L \subseteq L_-$. Since $L \not\subseteq L_-$, it is impossible. □

Corollary 2.14[75]. *Let $\mathcal{L}$ be a subspace lattice on H. If $\mathcal{L}$ satisfies one of the following conditions*

(1) $\vee\{K : K \in \mathcal{J}_{\mathcal{L}}\} = H$ *and for each* K *in* $\mathcal{J}_{\mathcal{L}}$, $K_- \vee K = H$,

(2) $\cap\{K : K \in \mathcal{J}_{\mathcal{L}}\} = 0$ *and for each* K *in* $\mathcal{J}_{\mathcal{L}}$, $K_- \vee K = H$,

then $alg\mathcal{L}$ *is strongly rank decomposable.*

If $\mathcal{L}$ is a completely distributive subspace lattice, by [80] we have $\vee\{L : L \in \mathcal{J}_{\mathcal{L}}\} = H$ and $\cap\{L_- : L \in \mathcal{J}_{\mathcal{L}}\} = 0$. By Theorem 2.12 or Theorem 2.13, we have the following result.

Corollary 2.15[90]. *Let* $\mathcal{L}$ *be a finite distributive subspace lattice on* H *which satisfies* $\mathcal{J}_{\mathcal{L}} = \mathcal{P}_{\mathcal{L}}$. *Then* $alg\mathcal{L}$ *is strongly rank decomposable.*

Lemma 2.16. *Let* $\mathcal{N}$ *be a nest and let* $\phi : E \mapsto \tilde{E}$ *be an order homomorphism of* $\mathcal{N}$ *into itself. Let*

$$\mathcal{M} = \{T \in B(H) : (I - \tilde{E})TE = 0, \text{ for all } E \in \mathcal{N}\}$$

and

$$S(\mathcal{M}) = \{S \in B(H) : tr(RS) = 0 \text{ for every } R \in \mathcal{M} \text{ of rank at most one }\}.$$

Then $T \in \mathcal{S}(\mathcal{M})$ *if and only if* $T(E) \subseteq E_{\sim}$, *where* $E_{\sim} = \vee\{F \in \mathcal{N} : \tilde{F} \subset E\}$.

Proof. By [30, p220], it follows that $e \otimes f \in \mathcal{M}$ if and only if there is an $E \in \mathcal{N}$ such that $e \in E$ and $f \in (E_{\sim})^{\perp}$. Hence $tr(Se \otimes f) = (Se, f) = 0$ for all R in $\mathcal{M}$ of rank at most one if and only if $S(E) \subseteq E_{\sim}$. □

Let $\mathcal{N}$ is a nest. Define $\tilde{\phi} : E \mapsto E_{\sim}$, where $E_{\sim} = \vee\{F \in \mathcal{N} : \tilde{F} \subset E\}$. It is easy to check that $\tilde{\phi}$ is also an order homomorphism of $\mathcal{N}$. By Lemma 2.16, $\mathcal{S}(\mathcal{M}) = \{T \in B(H) : (I - \tilde{\phi}(E))TE = 0, \text{ for all } E \in \mathcal{N}\}$. By Lemma 1.2[30] , we know the following result is true.

Corollary 2.17. *If* $\mathcal{N}$ *and* $\mathcal{S}(\mathcal{M})$ *are as in Lemma* 2.16, *then* $\mathcal{S}(\mathcal{M})$ *has strong rank decomposabiliy.*

If $\mathcal{L}_1$ and $\mathcal{L}_2$ are subspace lattices, a mapping $\phi : \mathcal{L}_1 \to \mathcal{L}_2$ is called a *complete lattice homorphism* if

$$\phi(\vee a_\lambda) = \vee \phi(a_\lambda) \text{ and } \phi(\cap a_\lambda) = \cap \phi(a_\lambda)$$

for every non-empty family $\{a_\lambda\}$ of elements of $\mathcal{L}_1$.

Lemma 2.18. *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be subspace lattices on H and let $\phi : E \mapsto \tilde{E}$ be a complete lattice homomorphism of $\mathcal{L}_1$ onto $\mathcal{L}_2$. Define*

$$\mathcal{M} = \{T \in B(H) : (I - \tilde{E})TE = 0 \textit{ for all } E \in \mathcal{L}_1\}.$$

Then $e \otimes f \in \mathcal{M}$ if and only if there exists an M in $\mathcal{L}_1$ such that $f \in (M_-)^\perp$, $e \in \tilde{M}$.

Proof. Suppose $e \otimes f \in \mathcal{M}$. Let $\tilde{M} = \cap\{\tilde{E} \in \mathcal{L}_2 : e \in \tilde{E}\}$. Let $M = \cap\{E \in \mathcal{L}_1 : \phi(E) = \tilde{M}\}$. Since ϕ is a complete lattice homomorphism, it follows that $\phi(M) = \tilde{M}$. If $N \in \mathcal{L}_1, M \not\subseteq N$, then $\phi(N) \not\supseteq \phi(M) = \tilde{M}$. Hence $(n, f) = 0$ for any $n \in N$, and $(m, f) = 0$, for $n \in M_-$.

Conversely, suppose that there exists an $M \in \mathcal{L}_1$ such that $e \in \tilde{M}$ and $f \in (M_-)^\perp$. If $N \in \mathcal{L}_1$ and $N \supseteq M$, then $\tilde{N} \supseteq \tilde{M}$, $e \otimes f(N) \subseteq \tilde{M} \subseteq \tilde{N}$. If $N \in \mathcal{L}_1$ and $N \not\supseteq M$, then $e \otimes f(N) = 0$. Hence $e \otimes f \in \mathcal{M}$. □

Theorem 2.19. *Let $\mathcal{L}_1$, $\mathcal{L}_2$ and $\mathcal{M}$ be as in Lemma 2.18. Suppose that $\vee\{M : M \in \mathcal{J}_{\mathcal{L}_1}\} = H$ and $K_- \cap K = 0$, for any $K \in \mathcal{J}_{\mathcal{L}_1}$. Then $\mathcal{M}$ is strongly rank decomposable.*

Proof: Suppose that $T \in \mathcal{M} \cap F(H)$ and $T \neq 0$. Since $T \neq 0$ and $\vee\{M : M \in \mathcal{J}_{\mathcal{L}_1}\} = H$, it follows that there exists E in $\mathcal{J}_{\mathcal{L}_1}$ such that $TE \neq 0$. Choose $e \in E$ such that $Te \neq 0$. Using $T \in \mathcal{M}$, we have $Te \in \tilde{E}$. Since ϕ is a complete homomorphism of $\mathcal{L}_1$ onto $\mathcal{L}_2$, it follows that $\tilde{E} \cap \widetilde{E_-} = 0$. Hence there is f in $\widetilde{E_-}^\perp$ such that $f(Te) = 1$. Using $T \in \mathcal{M}$, we have $T^*\widetilde{E_-}^\perp \subseteq (E_-)^\perp$. Thus $T^*f \in (E_-)^\perp$. By Lemma 2.18 $(Te) \otimes (T^*f) \in \mathcal{M}$. Lemma

2.2 implies that T has a rank one summand in $\mathcal{M}$. Since T is any non-zero finite operator, it follows that $\mathcal{M}$ has strong rank decomposability. □

Theorem 2.20. *Let $\mathcal{L}_1, \mathcal{L}_2, \mathcal{M}$ be as in Lemma 2.18. Suppose that $\cap\{F_- : F \in \mathcal{J}_{\mathcal{L}_1}\} = 0$ and $F \vee F_- = H$ for any $F \in \mathcal{J}_{\mathcal{L}_1}$. Then $\mathcal{M}$ has strong rank decomposability.*

Since the proof is similar to the proof of Theorem 2.19, we leave it to the reader.

Lemma 2.21. *Let $\mathcal{L}$ be a subspace lattice. If $H = \vee\{K : K \in \mathcal{J}_{\mathcal{L}}\}$ and $K_- \vee K = H$ for any K in $\mathcal{J}_{\mathcal{L}}$, then $K_- \cap K = 0$.*

Proof. Suppose $K \in \mathcal{J}_{\mathcal{L}}$, $K_- \cap K \neq 0$. Since $(K \cap K_-)_- \subseteq K_- \neq H$, it follows that $K \cap K_- \in \mathcal{J}_{\mathcal{L}}$. By hypothesis, $(K \cap K_-) \vee (K \cap K_-)_- = H$. Hence $(K \cap K_-) \vee K_- = H$ and $K_- = H$. This is impossible, since $K \in \mathcal{J}_{\mathcal{L}}$. □

A complex unital Banach algebra $\mathcal{A}$ is *semi-simple* if and only if it has no non-zero left ideals consisting entirely of quasinilpotent elements. $\mathcal{A}$ is said to be *semi-prime* if it has no non-zero left ideal whose square is zero. Clearly, $\mathcal{A}$ is semi-simple implies $\mathcal{A}$ is semi-prime.

By Theorem 1[82], and Lemma 2.21, we can obtain the following result

Corollary 2.22. *Let $\mathcal{L}$ be a subspace lattice satisfying $\vee\{L : L \in \mathcal{J}_{\mathcal{L}}\} = H$. The following are equivalent.*

(1) alg$\mathcal{L}$ is semi-simple,

(2) alg$\mathcal{L}$ is semi-prime,

(3) for every $L \in \mathcal{J}_{\mathcal{L}}$, $L \cap L_- = 0$ and $L \vee L_- = H$,

(4) for every $L \in \mathcal{J}_{\mathcal{L}}$, $L \vee L_- = H$,

(5) for every $L \in \mathcal{J}_{\mathcal{L}}$, $L \cap L_- = 0$.

Remarks 1. In [82], Longstaff shows that (1) to (4) are equivalent.

2. Corollary 2.22 implies that condition $L_- \vee L = H$ in Corollary 2.14(1) can be replaced by any of the conditions (1) to (4) of Corollary 2.22.

1.3 Isomorphisms of reflexive algebras

Let $\mathcal{A}_1 \subseteq B(H_1)$ and $\mathcal{A}_2 \subseteq B(H_2)$ be algebras. An algebraic isomorphism ϕ from $\mathcal{A}_1$ onto $\mathcal{A}_2$ is said to be *spatial* (or *spatially induced*) if there exists an invertible operator $S \in B(H_1, H_2)$ such that $\phi(A) = SAS^{-1}$, for every $A \in \mathcal{A}_1$. A slightly weaker condition is that ϕ be *quasi-spatial*; in this case we drop the assumption that S be bounded but we require that S be a closed densely defined, injective linear transformation, from H_1 onto a dense subset of H_2, with the properties that

(1) if x belongs to the domain of S, then Ax belongs to the domain of S, for every $A \in \mathcal{A}_1$;

(2) if x belongs to the domain of S, then $\phi(A)Sx = SAx$, for every $A \in \mathcal{A}_1$.

In [97], Ringrose proves that if $alg\mathcal{N}_1$ and $alg\mathcal{N}_2$ are nest algebras, then every algebraic isomorphism from $alg\mathcal{N}_1$ onto $alg\mathcal{N}_2$ is spatially induced. In [37], Gilfeater and Moore partially improve the result of Ringrose, by proving that if $\mathcal{L}_i$ is a completely distributive commutative subspace lattice, then every rank-preserving algebraic isomorphism from $alg\mathcal{L}_1$ onto $alg\mathcal{L}_2$ is quasi-spatially induced. Panaia [89] proves that if $\mathcal{L}_i$ is a finite distributive subspace lattice, then every rank-preserving algebraic isomorphism of $alg\mathcal{L}_1$ onto $alg\mathcal{L}_2$ is quasi-spatially induced. In [57], an example is given of an algebraic isomorphism between two identical algebras determined by an atomic Boolean subspace lattice for which the algebraic isomorphism is not spatially induced.

In this section, we prove that if $\mathcal{L}$, $\mathcal{L}_i$ $(i = 1, 2)$ are $\mathcal{J}$-subspace lattices, then every non-zero single element of $alg\mathcal{L}$ is rank-one, and any algebraic isomorphism between $alg\mathcal{L}_1$

and $alg\mathcal{L}_2$ is quasi-spatially induced. If $\mathcal{L}$ is a reflexive and $\vee$-distributive subspace lattice such that $H = \vee\{M : M \in \mathcal{J}_\mathcal{L}\}$, we also prove $(alg\mathcal{L})'' = alg\mathcal{F}$, where $\mathcal{F}$ is atomic Boolean.

Let $\mathcal{L}$ be a subspace lattice. The importance of $\mathcal{J}_\mathcal{L}$ lies in its intimate relation with the set of rank-one operators in $alg\mathcal{L}$. The following lemma is due to Longstaff, and we will use it repeatedly.

Lemma 3.1[80]. *Let $\mathcal{L}$ be a subspace lattice. Then the rank-one operator $e \otimes f$ belongs to $alg\mathcal{L}$ if and only if there is L in $\mathcal{L}$ such that $e \in L$ and $f \in L_-^\perp$.*

Definition 3.2 A subspace lattice $\mathcal{L}$ is called a $\mathcal{J}$-subspace lattice if

(1) $\vee\{L : L \in \mathcal{J}_\mathcal{L}\} = H$,

(2) $\cap\{L_- : L \in \mathcal{J}_\mathcal{L}\} = 0$,

(3) $L \vee L_- = H$, for any $L \in \mathcal{J}_\mathcal{L}$,

(4) $L \cap L_- = 0$, for any $L \in \mathcal{J}_\mathcal{L}$.

The class of $\mathcal{J}$-subspace lattices is rich. Every atomic Boolean subspace is a $\mathcal{J}$-subspace lattice and for any $\mathcal{J}$-subspace lattice both $\mathcal{L}^\perp$ and $latalg\mathcal{L}$ are $\mathcal{J}$-subspace lattices. The non-distributive pentagon subspace lattice is also in the class. We know that an atomic Boolean lattice is determined by its atoms and a $\mathcal{J}$-subspace lattice is not; different $\mathcal{J}$-subspaces can have the same sets of atoms. In [84], the connections between $\mathcal{J}$-subspace lattices and M-bases are studied

Let $\mathcal{L}$ be a $\mathcal{J}$-lattice and let $\{M_r\}_{r\in\Gamma}$ be its set of atoms. Then $\mathcal{J}_\mathcal{L} = \{M_r\}_{r\in\Gamma}$. Let I and J be disjoint subsets of Γ. Then

$$(\vee_{r\in I} M_r) \cap (\vee_{r\in J} M_r) = 0.$$

Definition 3.3. An element T of $alg\mathcal{L}$ is called single if whenever $ATB = 0$ with

$A, B \in \mathcal{A}$, then $AT = 0$ or $TB = 0$.

Lemma 3.4[73]. *Let $\mathcal{L}$ be a subspace lattice such that $\cap\{M_- : M \in \mathcal{J}_\mathcal{L}\} = 0$ and $\vee\{M : M \in \mathcal{J}_\mathcal{L}\} = H$. If T is single in $alg\mathcal{L}$, then there exists an $M \in \mathcal{J}_\mathcal{L}$ such that $T|_M \neq 0$ and $T|_M$ is an rank-one operator.*

Theorem 3.5. *Let $\mathcal{L}$ be a $\mathcal{J}$-subspace lattice. Then a non-zero element T in $alg\mathcal{L}$ is single if and only if $rankT = 1$.*

Proof. Let T be a non-zero single element in $alg\mathcal{L}$. By Lemma 3.4, there exists an M in $\mathcal{J}_\mathcal{L}$ such that $T|_M \neq 0$ and $T|_M$ is rank-one. For any $L \in \mathcal{J}_\mathcal{L}$ and $L \neq M$, we will show that $T(H) \subseteq L_-$. Since M is an atom, it follows that $M \subseteq L_-$, and $T(M) \subseteq M \subseteq L_-$. Let $f \in L_-^\perp$ be arbitrary. Choose $e \in L$, $e \neq 0$ and $m \in M_-^\perp$, $m \neq 0$. Let $n \in M$ with $Tn \neq 0$. By Lemma 3.1, $e \otimes f$ and $n \otimes m$ belong to $alg\mathcal{L}$. Since $Tn \in L_-$ and $f \in L_-^\perp$ it follows that

$$(e \otimes f)T(n \otimes m) = (Tn, f)e \otimes m = 0. \tag{3.1}$$

Since T is single and $T(n \otimes m) \neq 0$, by (3.1), it follows that $(e \otimes f)T = 0$. Hence for any x in H,

$$(Tx, f)e = (e \otimes f)Tx = 0. \tag{3.2}$$

By (3.2), we have that

$$Tx \in (L_-^\perp)^\perp = L_-. \tag{3.3}$$

By (3.3), it follows that $T(H) \subseteq \cap\{L_- : L \neq M, L \in \mathcal{J}_\mathcal{L}\}$.

Let K be an atom and $K \neq M$. In the following, we show that

$$K \cap \{L_- : L \neq M, L \in \mathcal{J}_\mathcal{L}\} = 0. \tag{3.4}$$

Suppose that this is not true. Then

$$K \cap \{L_- : L \neq M, L \in \mathcal{J}_{\mathcal{L}}\} = K. \tag{3.5}$$

Since $K \neq M$ and K is an atom, (3.5) implies that for any $L \neq M$, $K \subseteq L_-$. Thus $K \subseteq K_-$. This contradicts the fact that $K \cap K_- = 0$. For any $K \neq M$, $K \in \mathcal{J}_{\mathcal{L}}$, we have that $T(K) \subseteq K$ and $T(K) \subseteq T(H) \subseteq \cap\{L_- : L \neq M, L \in \mathcal{J}_{\mathcal{L}}\}$. By (3.4), it follows that $T(K) = 0$. Hence $T(span\{L : L \in \mathcal{J}_{\mathcal{L}}\}) \subseteq T(M)$. Since T is continuous and $\vee\{L : L \in \mathcal{J}_{\mathcal{L}}\} = H$, it follows that $T(H) \subseteq T(M)$. Thus T is a rank-one operator.

The converse is obvious. □

If $\mathcal{L}$ is a subspace lattice on H, the *ordered product* $\mathcal{L} \geq \mathcal{L}$ is the set of subspaces of $H \oplus H$ given by $\{L \oplus M : L, M \in \mathcal{L}, M \subseteq L\}$.

Lemma 3.6[73]. *Let $\mathcal{L}$ be a subspace lattice as in Lemma 3.4. Then every non-zero single element of $alg(\mathcal{L} \geq \mathcal{L})$ has rank-one if and only if every non-zero single element of $alg\mathcal{L}$ has rank-one.*

The following result is an easy conseqence of Theorem 3.5 and Lemma 3.6.

Corollary 3.7. *If $\mathcal{L}$ is a $\mathcal{J}$-lattice, then the non-zero element T of $alg(\mathcal{L} \geq \mathcal{L})$ is single if and only if rank $T = 1$.*

Theorem 3.8. *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be $\mathcal{J}$-lattices. If ϕ is an algebraic isomorphism of $alg\mathcal{L}_1$ onto $alg\mathcal{L}_2$, then ϕ is quasi-spatially induced.*

Using Theorem 3.8 and an argument similar to the proof of Theorem 7.2[37], we can prove the following result.

Corollary 3.9. *Let $\mathcal{L}$ be as in Theorem 3.8 and let δ be a derivation on $alg\mathcal{L}$. Then there is a linear transformation T such that $\delta(A)x = TAx - ATx$, for any $x \in span\{M$:*

$M \in \mathcal{J}_{\mathcal{L}}\}$.

To prove Theorem 3.8, we need several lemmas. By Theorem 3.5, we have

Lemma 3.10. *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be as in Theorem 3.8. Let ϕ be an algebraic isomorphism of $alg\mathcal{L}_1$ onto $alg\mathcal{L}_2$. Then ϕ maps rank-one operators in $alg\mathcal{L}_1$ onto rank-one operators in $alg\mathcal{L}_2$.*

Let $\mathcal{L}_i, i = 1, 2$ be $\mathcal{J}$-subspace lattices and ϕ be an algebraic isomorphism of $alg\mathcal{L}_1$ onto $alg\mathcal{L}_2$. For any $K \in \mathcal{J}_{\mathcal{L}_1}$, since $K \not\subseteq K_-$, we can choose $f_K \in K$ and $e_K \in K_-^{\perp}$ such that $(f_K, e_K) = 1$. Since ϕ is an algebraic isomorphism from $alg\mathcal{L}_1$ onto $alg\mathcal{L}_2$ and $f_K \otimes e_K$ is a single element of $alg\mathcal{L}_1$, it follows that $\phi(f_K \otimes e_K)$ is a single element of $alg\mathcal{L}_2$. By Theorem 3.5, $\phi(f_K \otimes e_K)$ is an rank-one operator in $alg\mathcal{L}_2$, and thus is of the form of $b_K \otimes a_K$. Let $L = \cap\{M \in \mathcal{L}_2 : b_K \in M\}$. Then $b_K \in L$. In fact $a_K \in L_-^{\perp}$ and $L \in \mathcal{J}_{\mathcal{L}_2}$. If $M \in \mathcal{L}_2, M \not\supseteq L$, then $b_K \notin M$. Since $b_K \otimes a_K \in alg\mathcal{L}_2$, it follows that for any $x \in M, b_K \otimes a_K(x) = 0$. Hence $a_K \in M^{\perp}$. Since $L_-^{\perp} = \cap\{M^{\perp} : M \not\supseteq L, M \in \mathcal{L}_2\}$, $a_K \in L_-^{\perp}$. Finally, since $(f_K \otimes e_K)^2 = f_K \otimes e_K$, $(b_K \otimes a_K)^2 = a_K \otimes b_K$ and $(b_K, a_K) = 1$.

Lemma 3.11. *Let K, L, e_K, f_K, a_K and b_K be given as above. Then the map $S_K : x \mapsto \phi(x \otimes f_K)b_K$ is a linear bijection of K onto L and $\{S_K K : K \in \mathcal{J}_{\mathcal{L}_1}\} = \mathcal{J}_{\mathcal{L}_2}$.*

Proof. It is clear that S_K is linear. The relation

$$\begin{aligned}(S_K x) \otimes a_K &= (\phi(x \otimes e_K)b_K) \otimes a_K = \phi(x \otimes e_K)(b_K \otimes a_K) \\ &= \phi(x \otimes e_K)\phi(f_K \otimes e_K) = \phi(x \otimes e_K),\end{aligned}$$

implies that if $x \in K$, $S_K x = 0$, then $\phi(x \otimes e_K) = 0$. Since ϕ is an algebraic isomorphism, it follows that $x = 0$, and S_K is injective on K. Now we prove that S_K is surjective. For any $l \in L$ and $l \neq 0$, Lemma 3.1 implies that $l \otimes a_K \in alg\mathcal{L}_2$. Let $\phi^{-1}(l \otimes a_K) = v \otimes u \in alg\mathcal{L}_1$.

Since $l \otimes a_K = (l \otimes a_K)(a_K \otimes b_K)$, it follows that $v \otimes u = (v \otimes u)(f_K \otimes e_K) = (f_K, u)v \otimes e_K$. Hence $(f_K, u) \neq 0$. Since $v \otimes u(K) \subseteq K$, it follows that $v \in K$. If $x = (f_K, u)v$, then $S_K x = \phi(x \otimes e_K)b_K = \phi(v \otimes u)b_K = (l \otimes a_K)b_K = l$.

It is obvious that $\{S_K K : K \in \mathcal{J}_{\mathcal{L}_1}\} \subseteq \mathcal{J}_{\mathcal{L}_2}$. Conversely let $L \in \mathcal{J}_{\mathcal{L}_2}$. Choose $a_K \in L_-^\perp$ and $b_K \in L$ such that $(b_K, a_K) = 1$. Since ϕ^{-1} is an algebraic isomorphism from $alg\mathcal{L}_2$ onto $alg\mathcal{L}_1$, an argument similar to the argument for ϕ proves that there exists K in $\mathcal{J}_{\mathcal{L}_1}$ such that $\phi^{-1}(b_K \otimes a_K) = f_K \otimes e_K$ with $e_K \in K_-^\perp$, $f_K \in K$ and $(f_K, e_K) = 1$. For this K, we have that S_K is a linear bijection from K onto L. □

Lemma 3.12. *Let K, L, e_k, f_k, a_K, b_K be as in Lemma* 3.11. *Then the map $T_K : x \mapsto (\phi(f_K \otimes x))^* a_K$ is a linear bijection of $K_-^\perp$ onto $L_-^\perp$ and $\{T_K(K_-^\perp) : K \in \mathcal{J}_{\mathcal{L}_1}\} = \{L_-^\perp : L \in \mathcal{J}_{\mathcal{L}_2}\}$.*

Proof. It is obvious that T_K is linear and $T_K(K_-^\perp) \subseteq L_-^\perp$. For any $x \in K_-^\perp$, the relation

$$b_K \otimes (T_K x) = (\phi(f_K \otimes x))^* b_K \otimes a_K = (b_K \otimes a_K)\phi(f_K \otimes x) = \phi(f_K \otimes x),$$

implies that $\phi(f_K \otimes x) = 0$, if $T_K x = 0$. Hence T_K is injective. For any $l \in L_-^\perp$, $l \neq 0$, let $\phi^{-1}(b_K \otimes l) = v \otimes u$ and let $y = (v, e_K)u$. Then $T_K y = l$ and $y \in K_-^\perp$.

It is clear that $\{T_K(K_-^\perp) : K \in \mathcal{J}_{\mathcal{L}_1}\} \subseteq \{L_-^\perp : L \in \mathcal{J}_{\mathcal{L}_2}\}$.

Conversely, for any $L \in \mathcal{J}_{\mathcal{L}_2}$, choose $b \in L$, and $a \in L_-^\perp$ such that $(b, a) = 1$. By Lemma 3.11, there exists K in $\mathcal{J}_{\mathcal{L}_1}$ such that $\phi^{-1}(b \otimes a) = f_K \otimes e_K$, $e_K \in K_-^\perp$ and $f_K \in K$ with $(f_K, e_K) = 1$. We have that $T_K(K_-^\perp) = L_-^\perp$. □

Lemma 3.13. *Let S_K and T_M be as in Lemmas* 3.11 *and* 3.12 *and let $K_i \in \mathcal{J}_{\mathcal{L}_1}$, $i = 1, .., t$. Then for any $m \in M_-^\perp$, $(\sum_{i=1}^{t} S_{K_i} x_i, T_M m) = (x, m)$, where $x = \sum_{i=1}^{t} x_i$, $x_i \in K_i$.*

Proof. Suppose that $S_M(M) = N$. By Lemma 3.12, $T_M(M_-^\perp) = N_-^\perp$. If $K_i \neq M$, for any $i = 1, ..., n$, then Lemma 3.11 implies $S_{K_i}(K_i) \subseteq N_-$. Thus $(\sum_{i=1}^{n} S_{K_i}x_i, T_M m) = 0 = (x, m)$. If there exists i_0 such that $K_{i_0} = M$ (say $i_0 = 1$), then

$$(\sum_{i=1}^{n} S_{K_i}x_i, T_M m) = (S_{K_1}x_1, T_{K_1}m) = (x_1 \otimes \phi(e_{K_1})b_{K_1}, (\phi(f_{K_1} \otimes m))^* a_{K_1})$$

$$= (\phi((f_{K_1} \otimes m)(x_1 \otimes e_{K_1})b_{K_1}, a_{K_1}) = (\phi(f_{K_1} \otimes e_{K_1})b_{K_1}, a_{K_1})(x_1, m) = (x_1, m). \square$$

Let $\hat{H} = span\{M : M \in \mathcal{J}_{\mathcal{L}_1}\}$. For any x in $\hat{H}$, $x = \sum_{j=1}^{p} x_j$ with $x_j \in M_j$ and $M_j \in \mathcal{J}_{\mathcal{L}_1}$. Let $Sx = \sum_{j=1}^{p} S_{M_j}x_j$.

Lemma 3.14. *Let $S : span\{M : M \in \mathcal{J}_{\mathcal{L}_1}\} \to span\{N : N \in \mathcal{J}_{\mathcal{L}_2}\}$ be the linear map as defined above. Then S has a closed extension.*

Proof. Let $x = \sum_{i=1}^{n} x_i$, $x_i \in M_i$ and $M_i \in \mathcal{J}_{\mathcal{L}_1}$. Suppose that $Sx = 0$. By Lemma 3.11, we have that if $i \neq j$, $S_{M_i} \cap S_{M_j} = 0$. Since S_{M_i} is injective, it follows that S is also injective. Let $(0, u)$ be in the closure of the graph of S. Let $\{x_n\} \subseteq span\{M : M \in \mathcal{J}_{\mathcal{L}_1}\}$ such that $(x_n, Sx_n) \to (0, u), n \to \infty$. Let $v \in span\{N_-^{\perp} : N \in \mathcal{J}_{\mathcal{L}_2}\}$. By Lemma 3.12, $v = \sum_{i=1}^{t} T_{M_i}m_i$, $M_i \in \mathcal{J}_{\mathcal{L}_1}, m_i \in M_{i-}^{\perp}$. By Lemma 3.13, $(Sx_n, T_{M_i}m_i) = (x_n, m_i)$, for $i = 1, ..., t$, so

$$\sum_{i=1}^{t}(x_n, m_i) = (Sx_n, \sum_{i=1}^{t} T_{M_i}m_i) = (Sx_n, v). \tag{3.6}$$

In (3.6), when n goes to infinity, we have $(u, v) = 0$ for any $v \in span\{N_-^{\perp} : N \in \mathcal{J}_{\mathcal{L}_2}\}$. Since $\mathcal{L}_2$ is a $\mathcal{J}$-subspace lattice, it follows that $u = 0$. $\square$

Let $\overline{S}$ denote the closed extension of S in Lemma 3.14. By (3.6), we have that if y belongs to the domain of $\overline{S}$ then $(\overline{S}y, T_M m) = (y, m)$, for any $m \in M$ and $M \in \mathcal{J}_{\mathcal{L}_1}$. Hence $\overline{S}$ is also injective.

Proof of Theorem 3.8.

To show that ϕ is quasi-spatially induced, we only need to prove that (i) the domain of $\overline{S}$ is an invariant linear manifold of $alg\mathcal{L}_1$, and (ii) $\phi(A)\overline{S}x = \overline{S}Ax$ for any x in the domain of $\overline{S}$. By the definition of S_K, we have that $S_K Ax = \phi(A)S_K x$, for any $x \in K$. Hence $SAx = \phi(A)Sx$, for any $x \in span\{K : K \in \mathcal{J}_{\mathcal{L}_1}\}$. For any x in $D(\overline{S})$, there exists $\{x_n\} \subseteq span\{K : K \in \mathcal{J}_{\mathcal{L}_1}\}$ such that $(x_n, Sx_n) \to (x, \overline{S}x)$. Since $SAx_n = \phi(A)Sx_n$ and $x_n \to x$, $Sx_n \to \overline{S}x$ we have that $Ax_n \to Ax$ and $SAx_n = \overline{S}Ax_n \to \phi(A)\overline{S}x$. Since $\overline{S}$ is closed, it follows that $Ax \in D(\overline{S})$ and $\overline{S}Ax = \phi(A)\overline{S}x$. □

In the remainder of the section, we consider the properties of double commutants of some reflexive algebras.

Proposition 3.15. *Let $\mathcal{L}$ be a subspace lattice such that $H = \vee\{M : M \in \mathcal{J}_{\mathcal{L}}\}$. Then $(alg\mathcal{L})'$ is abelian.*

Proof. Let $A, B \in (alg\mathcal{L})'$. By Corollary 2.3[72], for any $M \in \mathcal{J}_{\mathcal{L}}$, there exist scalars λ_A and λ_B such that $M \subseteq ker(A - \lambda_A I)$ and $M \subseteq ker(B - \lambda_B I)$. Hence for any $m \in M$, $ABm = BAm$. Since $H = \{M : M \in \mathcal{J}_{\mathcal{L}}\}$, it follows that $ABx = BAx$, for any $x \in H$. □

Corollary 3.16. *If $\mathcal{L}$ is a $\mathcal{J}$-subspace lattice, then $(alg\mathcal{L})'$ is abelian.*

A subspace lattice $\mathcal{L}$ is called $\vee$-*distributive* if $L \cap (\vee_{i\in I} L_i) = \vee_{i\in I}(L \cap L_i)$, for any index set I and any $L, L_i \in \mathcal{L}$. A subspace lattice $\mathcal{L}$ is said to be *completely distributive* if the following identity holds for arbitrary index sets:

$$\bigvee_{i\in A} (\bigwedge_{j\in B_i} L_{ij}) = \bigvee_{\psi\in\Pi B_i} (\bigwedge_{i\in A} L_{i\psi(i)}).$$

Theorem 3.17. *Let $\mathcal{L}$ be a reflexive subspace lattice such that $H = \vee\{M : M \in \mathcal{J}_{\mathcal{L}}\}$. Suppose that $\mathcal{L}$ is $\vee$-distributive. Then*

(1) $(alg\mathcal{L})'' = alg\mathcal{F}$, where $\mathcal{F}$ is an atomic Boolean lattice.

(2) For each finite set $L_1, ..., L_n$ of atoms of $\mathcal{F}$, $L_1 + ... + L_n$ is closed.

Proof. (1) By Corollary 2.3[72], for any $M \in \mathcal{J}_{\mathcal{L}}$ and for each $T \in (alg\mathcal{L})'$ there is a unique eigenspace M_T of T containing M. Define

$$K_M = \cap\{M_T : T \in (alg\mathcal{L})'\}.$$

By remark [72, p175], it follows that for any $M, N \in \mathcal{J}_{\mathcal{L}}$, either $K_M = K_N$ or $K_M \cap K_N = 0$. Let $\mathcal{F}$ be the subspace lattice generated by $0, H$ and all K_M for any M in $\mathcal{J}_{\mathcal{L}}$. By the hypotheses, $\mathcal{F}$ is $\vee$-distributive. By Proposition 3.2[84] and the fact that $\mathcal{F}$ is $\vee$-distributive, we have that $\mathcal{F}$ is a $\mathcal{J}$-subspace lattice. By Theorem 2.1[84], it follows that $\mathcal{F}$ is atomic and Boolean.

(2) We prove (2) by induction. For $n = 2$, by Proposition 3.2[84] and (1), there exist $M^i \in \mathcal{J}_{\mathcal{L}}, i = 1, 2$ such that $L_i = K_{M^i}, i = 1, 2$, where $M^i \in \mathcal{J}_{\mathcal{L}}$. Since $K_{M^i} = \cap\{{M^i}_T : T \in (alg\mathcal{L})'\}, i = 1, 2$, where M^i is the unique eigenspace of T containing M^i, and $L_1 \cap L_2 = 0$, it follows that there exist $T \in (alg\mathcal{L})'$ and distinct scalars λ, μ such that $L_1 \subseteq ker(T - \lambda I)$ and $L_2 \subseteq ker(T - \mu I)$. Let $P = (T - \lambda I)/(\mu - \lambda) \in (alg\mathcal{L})'$. Then $P|_{L_1} = 0, P|_{L_2} = I$, and therefore the sum of $L_1 + L_2$ is closed.

Let $L_1, ..., L_{n+1}$ be distinct atoms of $\mathcal{F}$. For the pairs $\{L_1, L_i\}, i = 2, ..., n+1$, there exist operators $P_i \in (alg\mathcal{L})'$ such that $P_i|_{L_1} = I$, $P_i|_{L_i} = 0, i = 2, ..., n+1$. Let $Q = P_1 \cdots P_{n+1}$. By Proposition 3.15, $(alg\mathcal{L})'$ is abelian. Thus this product is independent of the order of P_i. Thus $Q_i|_{L_1} = 0, i = 2, ..., n+1$ and $Q_i|_{L_1} = 1$. Hence $L_1 + \vee_{i=2}^{n+1} L_i$ is closed. By the induction hypothesis, $L_1 + ... + L_{n+1}$ is closed. □

Suppose that $\tilde{\mathcal{L}}$ is a commutative subspace on H and $\tilde{\mathcal{L}}$ is not completely distributive. Then $\tilde{\mathcal{L}}$ is $\vee$-distributive and reflexive. Let $\mathcal{L} = \{L \oplus 0 : L \in \tilde{\mathcal{L}}\} \cup \{H \oplus H\}$. Then $\mathcal{L}$ is

$\vee$-distributive and reflexive such that $H \oplus H = \{M : M \in \mathcal{J}_{\mathcal{L}}\}$. This shows that Theorem 3.17 improves Theorem 5.4[62].

Remark When I finished section 1.3, W. Longstaff told me that O. Panaia has independently proved Theorem 3.8.

Chapter 2

Boundedly Reflexive Subspaces and Applications

2.1 Boundedly reflexive subspaces of $B(H)$

In this section, we study a new type of reflexivity, which we call "bounded reflexivity", of a subspace of operators on a normed space. The concept of "bounded reflexivity" is implicitly contained in some papers, for instance, [50] and [76]. It plays an important role in the study of complete positivity of elementary operators (see [16]).

Let Y be a complex normed space and $B(Y)$ be the set of all bounded linear operators on Y, and $F(Y)$ the set of operators with finite rank. We use $F_n(Y)$ to denote the set of operators in $B(Y)$ with rank less than or equal to n. If $\mathcal{S}$ is a subspace of $B(Y)$, we let $\mathcal{S}_F = \mathcal{S} \cap F(Y)$. If $\mathcal{S}$ is a subset of $B(Y)$ and $r > 0$, define $\mathcal{S}_r = \{T \in \mathcal{S} : ||T|| \leq r\}$. Let $\mathcal{S}$ be a subspace of $B(Y)$, and let $ref_a(\mathcal{S}) = \{T \in B(Y) : Ty \in \mathcal{S}y, \text{ for all } y \in Y\}$. $\mathcal{S}$ is said to be reflexive if $ref(\mathcal{S}) = \mathcal{S}$ and $\mathcal{S}$ is said to be algebraically reflexive if $ref_a(\mathcal{S}) = \mathcal{S}$. Let $ref_b(\mathcal{S}) = \{T \in B(Y) : \text{ there exists } M_T \text{ such that } Ty \in [\mathcal{S}_{M_T}y], \text{ for all } y \in Y\}$, where $[\cdot]$ denotes the norm closure, and let $ref_{ab}(\mathcal{S}) = \{T \in B(Y) : \text{there exists an } M_T > 0 \text{ such that } Ty \in \mathcal{S}_{M_T}y, \text{ for all } y \in Y\}$. $\mathcal{S}$ is called boundedly reflexive if $\mathcal{S} = ref_b(\mathcal{S})$ and algebraically boundedly reflexive if $ref_{ab}(\mathcal{S}) = \mathcal{S}$. $\mathcal{S}$ is said to be boundedly (resp. algebraically boundedly) n-reflexive, if $\mathcal{S}^{(n)}$ is boundedly (resp. algebraically boundedly) reflexive.

Throughout this section, X denotes a complex Banach space. If $\mathcal{S}$ is a weakly closed subset of $B(X)$, then $[\mathcal{S}_M x] = \mathcal{S}_M x$. Thus $ref_{ab}(\mathcal{S}) = ref_b(\mathcal{S})$. For any $x \in X$, define the map $\phi_x : \mathcal{S} \to X$ by $\phi_x T = Tx$ for all $T \in \mathcal{S}$. A vector x is called a separating vector of $\mathcal{S}$ if ϕ_x is injective and x is called a strictly separating vector of $\mathcal{S}$ if ϕ_x is bounded below on $\mathcal{S}$. Let $M_{\mathcal{S}}$ be the set of strictly separating vectors of the subset $\mathcal{S}$ of $B(X)$. Then $M_{\mathcal{S}}$ is called linearly dense in X if $M_{\mathcal{S}}$ is nonempty and for any $x \in M_{\mathcal{S}}$ and $y \in X$, the set $G = \{\lambda \in \mathbf{C} : x + \lambda y \in M_{\mathcal{S}}\}$ is dense in the complex plane $\mathbf{C}$.

Let H be a separable complex Hilbert space and let $T(H)$ be the trace class operators. For any operator $T \in B(H)$, we use $\mathcal{W}(T)$ to denote the weakly closed algebra generated by T and the identity operator. Let $T \in B(H)$. T is said to be boundedly n-reflexive, if $\mathcal{W}(T^{(n)})$ is boundedly reflexive. For any subset $\mathcal{W}$ of $B(H)$, define $\mathcal{W}_0 = \{T \in T(H) : |tr(AT)| \leq 1, \text{ for all } A \in \mathcal{W}\}$. Similarly, for any subset of $\mathcal{V}$ of $T(H)$, we define $\mathcal{V}^0 = \{A \in B(H) : |tr(AT)| \leq 1, \text{ for all } T \in \mathcal{V}\}$. If $\mathcal{M}$ is a subset of $B(H)$, we let $\mathcal{M}_\perp = \{A \in T(H) : tr(AB) = 0 \text{ for all } B \in \mathcal{M}\}$.

2.1.1 Bounded reflexivity

The following proposition follows immediately from the definition of bounded reflexivity.

Proposition 1.1. *If $\mathcal{S}_i$ are subspaces of $B(X)$, for $i = 1, 2, \ldots$, then $\sum_{i=1}^{\infty} \oplus \mathcal{S}_i$ is boundedly n-reflexive in $B(\sum_{i=1}^{\infty} \oplus X_i)$ if and only if $\mathcal{S}_i$ is boundedly n-reflexive, i =1, 2,...*

It follows from the definitions that if $\mathcal{S}$ is reflexive then $\mathcal{S}$ is boundedly reflexive. The following example shows the converse is false. In fact, $\mathcal{S}$ can be boundedly reflexive and not n-reflexive for any n.

Example 1.2. Let

$$\mathcal{S}_n = \left\{ \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix} \in M_n(C) : \sum_{i=1}^{n} a_{ii} = 0 \right\}, \ n \geq 2.$$

Then $\mathcal{S}_n$ is boundedly reflexive.

Proof. The set of all $n \times n$ upper triangular matrices is reflexive. Thus if $T \in ref_b(\mathcal{S}_n)$, T must be an upper triangular matrix. Suppose that

$$T = \begin{pmatrix} t_{11} & t_{12} & \cdots & t_{1n} \\ 0 & t_{22} & \cdots & t_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & t_{nn} \end{pmatrix}$$

and there exists an $M_T > 0$ such that for any $x_k = (x_1^{(k)}, ..., x_n^{(k)})^t \in \mathbf{C}^{(n)}$, there exists

$$A_k = \begin{pmatrix} a_{11}^{(k)} & a_{12}^{(k)} & \cdots & a_{1n}^{(k)} \\ 0 & a_{22}^{(k)} & \cdots & a_{2n}^{(k)} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn}^{(k)} \end{pmatrix}$$

with $||A_k|| \leq M_T$ satisfying $Tx_k = A_k x_k$, or equivalently,

$$\begin{aligned} t_{11}x_1^{(k)} + t_{12}x_2^{(k)} + ... + t_{1n}x_n^{(k)} &= a_{11}^{(k)}x_1^{(k)} + a_{12}^{(k)}x_2^{(k)} + ... + a_{1n}^{(k)}x_n^{(k)}, \\ t_{22}x_2^{(k)} + ... + t_{2n}x_n^{(k)} &= a_{22}^{(k)}x_2^{(k)} + ... + a_{2n}^{(k)}x_n^{(k)}, \\ \cdots \quad & \quad \cdots \\ t_{nn}x_n^{(k)} &= a_{nn}^{(k)}x_n^{(k)}. \end{aligned}$$

Choose $x_i^{(k)}$ in such a way that $x_i^{(k)} \neq 0$, for any i, k and $\lim_{k\to\infty} |\frac{x_j^{(k)}}{x_i^{(k)}}| = 0$, for any $i < j$. Solving for t_{ii} from the above system of equations, we have $t_{nn} = a_{nn}^{(k)}$ and $t_{ii} = a_{ii}^{(k)} + \sum_{j=i+1}^{n} (a_{ij}^{(k)} - t_{ij}) \frac{x_j^{(k)}}{x_i^{(k)}}$, $1 \leq i < n$. Since $a_{ij}^{(k)}$ is bounded and $\lim_{k\to\infty} |\frac{x_j^{(k)}}{x_i^{(k)}}| = 0$, for any $i < j$, we have $\lim_{k\to\infty} a_{ii}^{(k)} = t_{ii}$, for any $1 \leq i \leq n$. Since $\sum_{i=1}^{n} a_{ii}^{(k)} = 0$ for any k, we have $\sum_{i=1}^{n} t_{ii} = 0$, that is $T \in \mathcal{S}_n$. Hence $\mathcal{S}_n$ is boundedly reflexive. However, $\mathcal{S}_n$ is not $(n-1)$-reflexive. Let $\mathcal{S} = \sum_{i=1}^{\infty} \oplus \mathcal{S}_i$. By Proposition 1.1, $\mathcal{S}$ is boundedly reflexive, but $\mathcal{S}$ is not n-reflexive for any n. □

From the previous example, one might be tempted to think, for any $n > 1$, could n-reflexivity imply bounded reflexivity? Although reflexivity implies bounded reflexivity, our next example shows that 2-reflexivity does not guarantee bounded reflexivity.

Example 1.3. Let

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \; E_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \; E_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

and $\mathcal{S} = span\{I, E_{12}, E_{21}\}$. Then $\mathcal{S}$ is 2-reflexive. However, $\mathcal{S}$ is not boundedly reflexive.

Proof. To see this, we only need to show

$$T = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in ref_b(\mathcal{S}).$$

For any $x = (x_1, x_2)^t \in \mathbf{C}^2$, it suffices to show we can find scalars t_1, t_2, and t_3 with $|t_i| \leq 1$, for $i = 1, 2, 3$ such that $Tx = t_1 Ix + t_2 E_{12} x + t_3 E_{21} x$, or equivalently

$$\begin{pmatrix} 0 \\ x_2 \end{pmatrix} = t_1 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + t_2 \begin{pmatrix} x_2 \\ 0 \end{pmatrix} + t_3 \begin{pmatrix} 0 \\ x_1 \end{pmatrix}.$$

If $x_2 = 0$, we choose $t_1 = 0$, $t_2 = 0$, and $t_3 = 0$. If $x_2 \neq 0$, and $|x_1| \leq |x_2|$, we choose

$t_1 = 1,\ t_2 = -\frac{x_1}{x_2}$, and $t_3 = 0$. If $x_2 \neq 0$, and $|x_1| > |x_2|$, we choose $t_1 = 0,\ t_2 = 0$, and $t_3 = \frac{x_2}{x_1}$. □

Theorem 1.4. *Suppose that $\mathcal{S}$ is a weakly closed subspace of $B(X)$ and $M_{\mathcal{S}}$ is the set of strictly separating vectors of $\mathcal{S}$. If $M_{\mathcal{S}}$ is linearly dense in H, then $\mathcal{S}$ is boundedly reflexive.*

Proof. By Proposition 2.3[9], $M_{\mathcal{S}}$ is an open set in H. Suppose that $T \in ref_b(\mathcal{S})$. For a fixed $x_0 \in M_{\mathcal{S}}$ and any $y \in H$, define $V = \{\lambda \in \mathbf{C} : x_0 + \lambda y \in M_{\mathcal{S}}\}$. For each $\lambda \in V$, let $H_{x,y}(\lambda)$ be the unique operator in $\mathcal{S}_{M_T}$ such that $T(x_0 + \lambda y) = H_{x,y}(\lambda)(x_0 + \lambda y)$. It follows from Proposition 2.8[9] that $H_{x,y}(\lambda)$ is analytic in V. Since $H_{x,y}(\lambda)$ is bounded in V and V is dense in $\mathbf{C}$, we can extend $H_{x,y}(\lambda)$ to a bounded analytic function in $\mathbf{C}$. Therefore, $H_{x,y}(\lambda)$ is a constant function. Thus there exists $A \in \mathcal{S}_{M_T}$ such that $T(x_0 + \lambda y) = A(x_0 + \lambda y)$ for all λ in $\mathbf{C}$. Let $\lambda = 0$, to get $Tx_0 = Ax_0$. Since x_0 is a separating vector of $\mathcal{S}$, A is unique. Since T nad A are linear, we get $Ty = Ay$. Since y is arbitrary, $T = A$, *i.e.*, $T \in \mathcal{S}$. □

A special case of the following corollary is proved in [50].

Corollary 1.5. *If $\mathcal{S}$ is a finite dimensional subspace of $B(X)$ and $\mathcal{S}$ has a separating vector, then $\mathcal{S}$ is boundedly reflexive. In particular, if $dim\mathcal{S} = n$ and every non-zero operator in $\mathcal{S}$ has rank greater than or equal to n, then $\mathcal{S}$ is boundedly reflexive.*

Proof. For any finite dimensional subspace of $B(X)$, all separating vectors are strictly separating vectors. By Proposition 4[39], the set of separating vectors of $\mathcal{S}$ is linearly dense. The conclusion follows. If $dim\mathcal{S} = n$ and every nonzero operator in $\mathcal{S}$ has rank greater than or equal to n, by Theorem 2[7], we have that $\mathcal{S}$ has a separating vector. Hence $\mathcal{S}$ is boundedly reflexive. □

Let $\mathcal{P}(t)$ denote the set of all complex polynomials and $\mathcal{P}(T) = \{P(T) : P(t) \in \mathcal{P}(t)\}$.

Corollary 1.6. *For every operator $T \in B(X)$, $\mathcal{P}(T)$ is algebraically boundedly reflexive.*

Proof. If T is not an algebraic operator, then $\mathcal{P}(T)$ is algebraically reflexive by Theorem 1[41], so $\mathcal{P}(T)$ is algebraically boundedly reflexive. If T is an algebraic operator, then $\mathcal{P}(T)$ is finite dimensional and has a separating vector. The conclusion now follows from Corollary 1.5. □

Next we give an example of an infinite dimensional subspace of $B(H)$ with a dense subset of H of strictly separating vectors.

Example 1.7. Let $H = l^2$ with the standard orthonormal basis $\{e_i\}_{i=1}^{\infty}$ and $K \in B(H)$ such that $\mathbf{C}\backslash\sigma(K)$ is dense in $\mathbf{C}$. Let $\mathcal{S}$ be the set of all bounded operators with matrix representions of the form $(x, Kx, 0, ..., 0...)$ for any $x \in H$, that is, x for the first column, Kx for the second column, and 0's for the other columns. Clearly $\mathcal{S}$ is weakly closed. Let $W = \{u = (u_1, u_2, u_3, ...)^t \in H : u_2 \neq 0,\ \frac{u_1}{u_2} \notin \sigma(K)\}$. One can check that all vectors in W are strictly separating vectors and the density of W in H follows from the fact that $\mathbf{C}\backslash\sigma(K)$ is dense in $\mathbf{C}$.

Next, we prove a theorem which provides an alternative description of bounded reflexivity.

Theorem 1.8. *Suppose that $\mathcal{S}$ is a subspace of $B(H)$ and $S_1 = \{T \in S : \|T\| \leq 1\}$. Then the following are equivalent.*

(1) $\mathcal{S}$ is boundedly n-reflexive,

(2) $((\mathcal{S}_1)_0 \cap F_n(H))^0 = \mathcal{S}_1$.

Proof. Suppose that (1) is true. It is obvious that $((\mathcal{S}_1)_0 \cap F_n(H))^0 \supseteq \mathcal{S}_1$. We only need to prove that $((\mathcal{S}_1)_0 \cap F_n(H))^0 \subseteq \mathcal{S}_1$. For any operator $T \in B(H)$ with $||T|| \leq 1$, if $T^{(n)} \notin \mathcal{S}_1^{(n)}$, then $T^{(n)} \notin ref_b(\mathcal{S}^{(n)})$. Therefore, there exists an $x_0 \in H^{(n)}$ such that $T^{(n)}x_0 \notin [\mathcal{S}_1^{(n)}x_0]$. Since $[\mathcal{S}_1^{(n)}x_0]$ is a convex and balanced set, there is a $y_0 \in H^{(n)}$ such that $Re(u, y_0) \leq 1 < (T^{(n)}x_0, y_0)$, for all $u \in [\mathcal{S}_1^{(n)}x_0]$. Equivalently, $|(A^{(n)}x_0, y_0)| \leq 1 < (T^{(n)}x_0, y_0)$, for all $A \in \mathcal{S}_1$. Let $x_0 = (x_1, ..., x_n)^t$, $y_0 = (y_1, ..., y_n)^t$ and $U = \sum_{i=1}^{n} x_i \otimes y_i$. We have that $|tr(AU)| = |tr(A^{(n)}(x_0 \otimes y_0))| \leq 1$, for any $A \in \mathcal{S}_1$, and therefore $U \in (\mathcal{S}_1)_0 \cap F_n(H)$. Since $tr(TU) = (T^{(n)}x_0, y_0) > 1$, $T \notin ((\mathcal{S}_1)_0 \cap F_n(H))^0$.

Suppose that (2) is true, and let $T \in ref_b(\mathcal{S}^{(n)})$. Since $ref_b(\mathcal{S}^{(n)}) \subseteq B(H)^{(n)}$, it follows that $T = U^{(n)}$ for some $U \in B(H)$. For any $T = U^{(n)} \in ref_b(\mathcal{S}^{(n)})$, there exists a nonzero scalar λ such that $\lambda Tx \in [\mathcal{S}_1^{(n)}x]$, for any $x \in H^{(n)}$. Hence for any $x,\ y \in H^{(n)}$,

$$(\lambda Tx, y) \in \overline{\{(S^{(n)}x, y) : S \in \mathcal{S}_1\}}. \tag{1.1}$$

For any $V \in (\mathcal{S}_1)_0 \cap F_n(H)$, let $V = \sum_{i=1}^{n} x_i \otimes y_i$, $x = (x_1, ..., x_n)^t$, $y = (y_1, ..., y_n)^t \in H^{(n)}$. We have that

$$|(S^{(n)}x, y)| = |tr(SV)| \leq 1$$

for any $S \in \mathcal{S}_1$. Relation (1.1) implies that $|tr(\lambda TV)| \leq 1$. This implies $\lambda T \in ((\mathcal{S}_1)_0 \cap F_n(H))^0 = \mathcal{S}_1$, thus $T \in \mathcal{S}$. □

Corollary 1.9 is an immediate consequence of the above theorem.

Corollary 1.9. *A subspace $\mathcal{S}$ of $B(H)$ is boundedly reflexive if and only if $\mathcal{S}^*$ is boundedly reflexive.*

Corollary 1.10. *A subspace $\mathcal{S}$ of $B(H)$ is boundedly reflexive if and only if $\mathcal{S}_1$ is w^*-closed and $\mathcal{S}$ is algebraically boundedly reflexive.*

Remark We can view (2) of Theorem 1.8 as an alternative definition of bounded reflexivity. One advantage of doing this is that it enables us easily to adapt the techniques used in [10] to construct counterexamples. It follows also from Theorem 1.8 that if $\mathcal{S}$ is boundedly n-reflexive, then $\mathcal{S}$ is boundedly m-reflexive for $n \leq m$ and $\mathcal{S}$ is w^*-closed.

Using Theorem 1.8, we can prove the following result about bounded reflexivity of direct integrals. Since the proof involves many definitions and notation, we omit it.

Proposition 1.11. *Let (Λ, Ω, μ) be a complete σ-finite measure space. Suppose that $\{\varphi_\omega : \omega \in \Lambda\}$ is a measurable families of w^*-closed linear subspaces of $B(H)$. Then $ref_b(\int_\Lambda^\oplus \varphi_\omega d\mu(\omega)) = \int_\Lambda^\oplus ref_b(\varphi_\omega) d\mu(\omega)$.*

Theorem 1.12. *Let $T \in T(H)$ and $\mathcal{S} = \{A \in B(H) : tr(AT) = 0\}$. Then the following are equivalent.*

(1) $rankT \leq n$,

(2) $\mathcal{S}$ is n-reflexive,

(3) $\mathcal{S}$ is boundedly n-reflexive.

Proof. It is obvious that (1) implies (2) and (2) implies (3).

Suppose (3) is true. We will prove (1) is true. We only need to show that if $rankT \geq m > n$, then $\mathcal{S}$ is not boundedly $(m-1)$-reflexive. Hence $\mathcal{S}$ is not n-reflexive.

Suppose $m > n$. Choose invertible operators $U,\ V \in B(H)$ such that UTV has an operator matrix of the form $\begin{pmatrix} I_m & B \\ C & D \end{pmatrix}$ relative to $H = \mathbf{C}^{(m)} \oplus H_1$. Since

$$V^{-1}\mathcal{S}U^{-1} = \{A \in B(H) : tr(\begin{pmatrix} I_m & B \\ C & D \end{pmatrix} A) = 0\},$$

we may assume that

$$T = \begin{pmatrix} I_m & B \\ C & D \end{pmatrix}.$$

Note $(m+2)T \in (\mathcal{S}_1)_0$. Theorem 1.8 now shows that to prove that $\mathcal{S}$ is not boundedly $(m-1)$-reflexive, it is enough to prove

$$(m+2)T \notin \overline{co((\mathcal{S}_1)_0 \cap F_{m-1}(H))}^{\|\cdot\|_1}.$$

Suppose that $(m+2)T \in \overline{co((\mathcal{S}_1)_0 \cap F_{m-1}(H))}^{\|\cdot\|_1}$. Choose $\lambda_i > 0$ with $\sum_{i=1}^{t} \lambda_i = 1$ and $A_i \in (\mathcal{S}_1)_0 \cap F_{m-1}(H)$ satisfying

$$\|\sum_{i=1}^{t} \lambda_i A_i - (m+2)T\|_1 < 1. \tag{1.2}$$

Let $\tilde{E}_{ij}$ denote the $m \times m$ matrix with 1 in (i,j) place and zeros elsewhere. Let

$$E_{ij} = \begin{pmatrix} \tilde{E}_{ij} & 0 \\ 0 & 0 \end{pmatrix}$$

be an operator matrix relative to $H = \mathbf{C}^{(m)} \oplus H_1$. Relation (1.2) implies

$$\|\sum_{i=1}^{t} \lambda_i E_{11} A_i E_{11} - (m+2)E_{11}TE_{11}\|_1 < 1. \tag{1.3}$$

Since $\sum_{i=1}^{t} \lambda_i = 1,\ \lambda_i > 0$, by (1.3), there exists an i_0 such that $\|E_{11}A_{i_0}E_{11}\| \geq m+1$. Let

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

be a matrix representation of A_{i_0} relative to $H = \mathbf{C}^{(m)} \oplus H_1$, where $A_{11} = (a_{ij})_{m\times m}$. Then $|a_{11}| \geq m+1$. Since $E_{ij} \in \mathcal{S}_1$ for $i \neq j$ and $A_{i_0} \in (\mathcal{S}_1)_0$, it follows that $|tr(E_{ij}A_{i_0})| \leq 1$, and therefore

$$|a_{ji}| \leq 1 \text{ for } i \neq j. \tag{1.4}$$

Similarly, since $E_{11} - E_{ii} \in \mathcal{S}_1$ for $i = 2, ..., m$, we have $|tr((E_{11} - E_{ii})A_{i_0})| \leq 1$, and therefore

$$|a_{11} - a_{ii}| \leq 1, \text{ for } i = 1, ..., m. \tag{1.5}$$

Relation (1.5) and the fact $|a_{11}| \geq m + 1$ imply

$$|a_{ii}| \geq m, \text{ for } i = 2, ..., m. \tag{1.6}$$

By (1.4), we obtain

$$\sum_{\substack{j=1 \\ j \neq i}}^{m} |a_{ij}| \leq m - 1, \text{ for } i = 1, ..., m. \tag{1.7}$$

By (1.6) and (1.7), it follows that

$$|a_{ii}| > \sum_{\substack{j=1 \\ j \neq i}}^{m} |a_{ij}| \leq m - 1, \text{ for } i = 1, ..., m.$$

i.e., A_{11} is diagonal-dominant. So A_{11} is invertible and $rank A_{11} = m$. This contradicts the fact that $rank A_{i_0} \leq m - 1$. □

Remark Theorem 1.12 provides a simple way of constructing subspaces of $B(H)$ that are boundedly $(n+1)$-reflexive, but not boundedly n-reflexive. Theorem 1.12 also generalizes Proposition 1.3[42].

The next corollary is an easy consequence of Theorem 1.12.

Corollary 1.13. *Suppose that $\mathcal{S}$ is a subspace of $M_n(\mathbf{C})$ with $dim\mathcal{S} = n^2 - 1$. Then $\mathcal{S}$ is boundedly k-reflexive if and only if $\mathcal{S}$ is k-reflexive.*

Definition 1.14. Suppose $\mathcal{S}$ is a w^*-closed (resp. weakly closed) subspace of $B(H)$. We say that $\mathcal{S}$ has the property W^*P_n (resp. WP_n) if for every w^*-closed (resp. weakly closed) subspace $\tilde{\mathcal{S}}$ of $\mathcal{S}$, for any $A \in \mathcal{S} \backslash \tilde{\mathcal{S}}$, and for any $M > 0$, there exists an operator

$T = \sum_{i=1}^{n} x_i \otimes y_i$ such that T separates A from $\tilde{S}_M$, *i.e.*,

$$\sum_{i=1}^{n} tr(A(x_i \otimes y_i)) \notin \overline{\{\sum_{i=1}^{n} tr(B(x_i \otimes y_i)) : B \in \tilde{S}_M\}},$$

or $\sum_{i=1}^{n}(Ax_i, y_i) \notin \overline{\{\sum_{i=1}^{n}(Bx_i, y_i) : B \in \tilde{S}_M\}}$.

Proposition 1.15. *Let $\mathcal{S}$ be a w^*-closed (resp. weakly closed) and boundedly n-reflexive subspace of $B(H)$. Then every w^*-closed (resp. weakly closed) subspace of $\mathcal{S}$ is boundedly n-reflexive if and only if $\mathcal{S}$ has property W^*P_n (resp. WP_n).*

Proof. We give a proof for the case that $\mathcal{S}$ is w^*-closed. The other case can be proved similarly.

Suppose that $\mathcal{S}$ has property W^*P_n. Let $\tilde{\mathcal{S}}$ be a w^*-closed subspace of $\mathcal{S}$, and let $A \notin \tilde{\mathcal{S}}$. We need to prove $A^{(n)} \notin ref_b(\tilde{\mathcal{S}}^{(n)})$. If $A \notin \mathcal{S}$, then $A^{(n)} \notin \mathcal{S}^{(n)} = ref_b(\mathcal{S}^{(n)})$. Then, clearly, $A^{(n)} \notin ref_b(\tilde{\mathcal{S}}^{(n)})$. Suppose $A \in \mathcal{S}\backslash\tilde{\mathcal{S}}$. Since $\mathcal{S}$ has property W^*P_n, we have, for any $M > 0$, there exists a $T = \sum_{i=1}^{n} x_i \otimes y_i$ such that $\sum_{i=1}^{n}(Ax_i, y_i) \notin \overline{\{\sum_{i=1}^{n}(Bx_i, y_i) : B \in \tilde{S}_M\}}$. This implies $A^{(n)}x \notin [\tilde{\mathcal{S}}_M^{(n)}x]$, where $x = (x_1, ..., x_n)^t$. Thus $A \notin ref_b(\tilde{\mathcal{S}})$.

Conversely, suppose that $\mathcal{S}$ does not have property W^*P_n. Then there exist a w^*-closed subspace $\tilde{\mathcal{S}}$ of $\mathcal{S}$, $A \in \mathcal{S}\backslash\tilde{\mathcal{S}}$, and $M > 0$ such that for any $T = \sum_{i=1}^{n} x_i \otimes y_i$, we have that $\sum_{i=1}^{n}(Ax_i, y_i) \in \overline{\{\sum_{i=1}^{n}(Bx_i, y_i) : B \in \tilde{S}_M\}}$. This implies $A^{(n)}x \in [\tilde{\mathcal{S}}_M^{(n)}x]$, for any $x = (x_1, ..., x_n)^t$. Therefore, $A^{(n)} \in ref_b(\tilde{\mathcal{S}}^{(n)})$, thus $\tilde{\mathcal{S}}$ is not boundedly reflexive. □

Proposition 1.16. *Let $\mathcal{S}$ be a subspace of $M_n(\mathbf{C})$, $n \geq 2$. If $dim\mathcal{S} \leq n^2 - 1$, then $\mathcal{S}$ has property W^*P_{n-1}.*

Proof. Let $\tilde{\mathcal{S}}$ be any subspace of $\mathcal{S}$. For any $A \in \mathcal{S}\backslash\tilde{\mathcal{S}}$, and $M > 0$, there exists an $R \in M_n(\mathbf{C})$ that separates A from $\tilde{S}_M$. Let T be in $\mathcal{S}_\perp$. Then for any scalar z, $zT + R$ separates A from $\tilde{S}_M$. If we choose any R_0 very close to R, then R_0 separates A from $\tilde{S}_M$

also. Choose such R_0 so that $det(zT + R_0)$ is not a constant function of z. Let z_0 be any solution of the equation $det(z_0T + R_0) = 0$. Then $z_0T + R_0$ separates A from $\tilde{S}_M$ and $rank(z_0T + R_0) \leq n - 1$. □

It is not hard to show that property P_n, or n-elementary, implies property W^*P_n. The following example shows that property W^*P_n for a subspace of $B(H)$ is weaker than property P_n, or n-elementary as defined in [8].

Example 1.17. Let E_{1n} be the $n \times n$ matrix, $n \geq 2$, with 1 in $(1, n)$ place and zeros elsewhere. Let $\mathcal{S} = \{A \in M_n(\mathbf{C}) : tr(E_{1n}A) = 0\}$. By Theorem 1.12, $\mathcal{S}$ is reflexive, and thus $\mathcal{S}$ is $(n-1)$-reflexive. Let $\tilde{\mathcal{S}}$ be the subspace $\mathcal{S}_n$ in Example 1.2. Clearly $\tilde{\mathcal{S}} \subseteq \mathcal{S}$, but $\tilde{\mathcal{S}}$ is not $(n-1)$-reflexive, and thus $\mathcal{S}$ does not have property P_{n-1}. However, $\mathcal{S}$ has property W^*P_{n-1} (resp. WP_{n-1}), by Proposition 1.16. □

Suppose that U and V are isometries acting between Hilbert spaces H and K. If $\mathcal{A}$ is a subset of $B(H)$ and $\mathcal{B}$ is a subset of $B(K)$ satisfying $U\mathcal{A}V^* \subseteq \mathcal{B}$ and $U^*\mathcal{B}V \subseteq \mathcal{A}$, then we say that $\mathcal{A}$ is a spatial direct summand of $\mathcal{B}$.

The next proposition follows directly from the definition of bounded reflexivity.

Proposition 1.18. *Suppose U and V are isometries from H into K and $\mathcal{S}$ is a subspace of $B(H)$. Then $U(ref_b(\mathcal{S}))V^* = ref_b(U\mathcal{S}V^*)$.*

The following is similar to Lemma 3.1[10].

Lemma 1.19. *Let $\mathcal{A}$ and $\mathcal{B}$ be subsets of $B(H)$ and $B(K)$, respectively. Suppose that U, V are isometries from H into K such that $U\mathcal{A}V^* \subseteq \mathcal{B}$ and $U^*\mathcal{B}V \subseteq \mathcal{A}$. Let $\mathcal{A}_1 = \{A \in \mathcal{A} : \ ||A|| \leq 1\}$ and $\mathcal{B}_1 = \{B \in \mathcal{B} : \ ||B|| \leq 1\}$. Let $G(H)$ stand for $T(H)$, $F(H)$, or $F_n(H)$ and $G(K)$ stand for $T(K)$, $F(K)$, or $F_n(K)$, respectively. Then*

(1) $U^*\mathcal{B}_1V = \mathcal{A}_1$,

(2) $V^*(\mathcal{B}_1)_0U = (\mathcal{A}_1)_0$,

(3) $(\mathcal{B}_1)_0 = V(\mathcal{A}_1)_0U^*$,

(4) $V^*((\mathcal{B}_1)_0 + G(K))U = (\mathcal{A}_1)_0 + G(K)$,

(5) $((\mathcal{A}_1)_0 \cap G(K))^0 = U^*((\mathcal{B}_1)_0 \cap G(K))^0V$,

(6) $\overline{co((\mathcal{A}_1)_0 \cap G(K))}^{\|\cdot\|_1} = V^*\overline{co((\mathcal{B}_1)_0 \cap G(K))}^{\|\cdot\|_1}U$, *where co denotes the convex hull.*

Proof. (1) Since $U\mathcal{A}_1V^* \subseteq \mathcal{B}_1$, it follows that $U^*U\mathcal{A}_1V^*V = \mathcal{A}_1 \subseteq U^*\mathcal{B}_1V$. Since $U^*\mathcal{B}_1V \subseteq \mathcal{A}_1$, we have $U^*\mathcal{B}_1V = \mathcal{A}_1$.

(2) If $T \in (\mathcal{B}_1)_0$, let $X = V^*TU$. For any $A \in \mathcal{A}_1$, we have $|tr(AX)| = |tr(AV^*TU)| = |tr(UAV^*T)| \leq 1$, so $V^*(\mathcal{B}_1)_0U \subseteq (\mathcal{A}_1)_0$. If $T \in (\mathcal{A}_1)_0$ let $Y = VTU^*$. For any $B \in \mathcal{B}_1$, we have $|tr(BY)| = |tr(BVTU^*)| = |tr(U^*BVT)| \leq 1$, *i.e.*, $V(\mathcal{A}_1)_0U^* \subseteq (\mathcal{B}_1)_0$. This implies $(\mathcal{A}_1)_0 \subseteq V^*(\mathcal{B}_1)_0U$.

(3) This part is contained in the proof of (2).

(4) Clearly, $V^*G(K)U \subseteq G(K)$ and $VG(K)U^* \subseteq G(K)$. Hence $V^*G(K)U = G(H)$. It follows from (2) that $V^*((\mathcal{B}_1)_0 + G(K))U = (\mathcal{A}_1)_0 + G(K)$.

(5) Let $X \in ((\mathcal{A}_1)_0 \cap G(K))^0$. Let $Y \in (\mathcal{B}_1)_0 \cap G(H)$. We have $V^*YU \in (\mathcal{A}_1)_0 \cap G(H)$. Therefore $|tr(UXV^*Y)| = |tr(XV^*YU)| \leq 1$, which implies $UXV^* \in ((\mathcal{B}_1)_0 \cap G(K))^0$. This shows $((\mathcal{A}_1)_0 \cap G(H))^0 \subseteq U^*((\mathcal{B}_1)_0 \cap G(K))^0V$.

The reverse inclusion is similar.

(6) To show " $\supseteq$ ", let P be the projection UU^* and Q be the projection VV^*. For any $X \in \overline{co((\mathcal{B}_1)_0 \cap G(K))}^{\|\cdot\|_1}$, take any $Y \in ((\mathcal{A}_1)_0 \cap G(K))^0$. By (5), $Y = U^*ZV$ for some $Z \in ((\mathcal{B}_1)_0 \cap G(K))^0$. Therefore, $|tr(YV^*XU)| = |tr(U^*ZVV^*XU)| = |tr(ZVV^*XUU^*)| =$

$|tr(ZQXP)| \leq |tr(ZQX)| = |tr(XZQ)| \leq |tr(XZ)| \leq 1$.

It remains to prove " $\subseteq$ ". For any $X \in \overline{co((\mathcal{A}_1)_0 \cap G(K))}^{\|\cdot\|_1}$ and $Y \in ((\mathcal{B}_1)_0 \cap G(K))^0$, by (5), $U^*YV \in ((\mathcal{A}_1)_0 \cap G(K))^0$. Therefore, $|tr(YVXU^*)| = |tr(U^*YVX)| \leq 1$, which implies $VXU^* \in \overline{co((\mathcal{B}_1)_0 \cap G(K))}^{\|\cdot\|_1}$. Thus $X \in V^*\overline{co((\mathcal{B}_1)_0 \cap G(K))}^{\|\cdot\|_1}U$. □

Proposition 1.20. *Suppose $\mathcal{A}$ and $\mathcal{B}$ are w^*-closed subspaces of $B(H)$ and $B(K)$, and $\mathcal{A}$ is a spatial direct summand of $\mathcal{B}$. If $\mathcal{B}$ is boundedly n-reflexive, then $\mathcal{A}$ is boundedly n-reflexive.*

Proof. If $\mathcal{B}$ is boundedly reflexive, then by Theorem 2.8, $\overline{co((\mathcal{B}_1)_0 \cap F_n(K))}^{\|\cdot\|_1} = (\mathcal{B}_1)_0$. By (2) and (6) of Lemma 1.19, $V^*(\mathcal{B}_1)_0U = (\mathcal{A}_1)_0 = \overline{co((\mathcal{A}_1)_0 \cap F_n(H))}^{\|\cdot\|_1} = V^*\overline{co((\mathcal{B}_1)_0 \cap F_n(K))}^{\|\cdot\|_1}U$. So $\mathcal{A}$ is boundedly reflexive. □

The following corollary answers a question from [42].

Corollary 1.21. *For any natural number k, there is a compact operator A such that $\mathcal{W}(A)$ is not boundedly k-reflexive.*

Proof. Let n be large enough and let $\mathcal{S}$ be a subspace of $M_n(\mathbf{C})$ such that $\mathcal{S}$ is not boundedly k-reflexive. By Proposition 1.1[10] and Theorem 2.3[10], there exist a compact operator A and isometries U, V satisfying

$$USV^* \subseteq \overline{\{P(A) : P \text{ is a polynomial with } P(0) = 0\}}^{\|\cdot\|},$$

$$U^*\overline{\{P(A) : P \text{ is a polynomial with } P(0) = 0\}}^{\|\cdot\|}V \subseteq \mathcal{S}.$$

By the proof of Theorem 2.3[10], we can choose $U^*V = 0$. Then we have that $\mathcal{W}(A) \supseteq USV^*$, $\mathcal{S} \supseteq U^*\mathcal{W}(A)V$. Now, Theorem 1.8 implies that $\mathcal{W}(A)$ is not boundedly k-reflexive. □

Based on [108], it is natural to ask the following questions about bounded reflexivity.

(1) Is $\mathcal{W}(A) = \{T\}' \cap ref_b(\mathcal{W}(T))$?

(2) Is $T \oplus T$ always boundedly reflexive ?

(3) If T_1 and T_2 are boundedly reflexive, is $T_1 \oplus T_2$ boundedly reflexive ?

Corollary 1.21 implies that (2) is not true. Since (1) implies (2), it follows that (1) is not true.

To answer question (3), we need the following lemma.

Lemma 1.22. *There exists a subspace $\mathcal{S}$ of $B(H \oplus H)$ so that $\mathcal{S}|_{H\oplus 0}$ and $\mathcal{S}|_{0\oplus H}$ are reflexive but $\mathcal{S}$ is not boundely n-reflexive.*

Proof. Let $\mathcal{M}$ be a subspace of $B(H)$ such that $\mathcal{M}$ is not boundedly n-reflexive. Define $\mathcal{S} = \{A \oplus B \in B(H \oplus H) : A - B \in \mathcal{M}\}$. Clearly, $\mathcal{S}|_{H\oplus 0} = B(H) \oplus 0$, $\mathcal{S}|_{0\oplus H} = 0 \oplus B(H)$, they are both reflexive.

Since $\mathcal{M}$ is not boundedly n-reflexive, we can choose that $T^{(n)} \in ref_b(\mathcal{M}^{(n)}) \backslash \mathcal{M}^{(n)}$. Let $U^{(n)} = T^{(n)} \oplus 0 \in B(H^{(n)} \oplus H^{(n)})$. Then $U^{(n)} \notin \mathcal{S}^{(n)}$ and $U^{(n)} \in ref_b(\mathcal{S}^{(n)})$. Hence $\mathcal{S}$ is not boundedly n-reflexive. □

Replacing Lemma 7[108] by Lemma 1.22 and using the same techniques as those in Example 7[108], we can construct the following example.

Example 1.23. If $1 \leq n < \infty$, then there are reflexive operators T_1 and T_2 such that $T_1 \oplus T_2$ is not boundedly n-reflexive.

2.1.2 Algebraic bounded reflexivity

It follows from Corollary 1.9 that a subspace $\mathcal{S}$ is boundedly reflexive if and only if $\mathcal{S}^*$ is boundedly reflexive. Our next example shows this is generally not true for algebraic bounded reflexivity.

Example 1.24. There exists an algebraically boundedly reflexive subspace $\mathcal{S}$ of $B(H)$ such that $\mathcal{S}^*$ is not algebraically boundedly reflexive.

Proof. Let $\{e_i\}_{i=1}^{\infty}$ be a orthonormal basis for a Hilbert space H, and let $\mathcal{S} = span\{e_i \otimes e_1 : i = 1, 2, ...\}$. Then $\mathcal{S}$ is algebraically boundedly reflexive and $\mathcal{S}^* = span\{e_1 \otimes e_i : i = 1, 2, ...\}$. We claim that $ref_{ab}(\mathcal{S}^*) = \{e_1 \otimes y : y \in H\}$. Clearly, $ref_{ab}(\mathcal{S}^*) \subseteq \{e_1 \otimes y : y \in H\}$. To see the reverse inclusion, take any x, $y \in H$ such that $||x|| = ||y|| = 1$. We show that $e_1 \otimes y(x) \in \mathcal{S}_2^* x$. Let $x_n = \sum_{i=1}^{n} (x, e_i) e_i$. Then $\lim_{n \to \infty} x_n = x$. Choose N so that $||x_N||^2 > 1/2$. Then there exists a t_N with $0 < |t_N| \leq 2$ so that $t_N ||x_N||^2 = (x, y)$. Define $A_N = t_N \sum_{i=1}^{N} (x, e_i) e_1 \otimes e_i$. Then $||A_N|| \leq 2$ and $A_N x = t_N ||x_N||^2 e_1 = (x, y) e_1 = e_1 \otimes y(x)$. Thus, $\mathcal{S}^*$ is not algebraically boundedly reflexive. □

Theorem 1.25. *Suppose that $\mathcal{S}$ is a linear subspace of $B(X)$ with a denumerable Hamel basis. Let $\mathcal{S}_I$ be any vector space complement of $\mathcal{S}_F$ in $\mathcal{S}$. Suppose that, for any subspace E of X with a denumerable Hamel basis, there exists a separating vector $y \in X$ for $\mathcal{S}_I$ such that $\mathcal{S}_I y \cap E = 0$ and $\mathcal{S}_F y$ is finite dimensional in X. Then $ref_{ab}(\mathcal{S}) = \mathcal{S} + ref_{ab}(\mathcal{S}_F)$. In this case, $\mathcal{S}$ is algebraically boundedly reflexive if and only if $\mathcal{S}_F$ is algebraically boundedly reflexive.*

Before proving Theorem 1.25, we give several corollaries of the result.

Corollary 1.26. *Let $\mathcal{S}$ and $\mathcal{S}_F$ be as in Theorem 1.25. If $\mathcal{S}_F X = span\{SX : S \in \mathcal{S}_F\}$ is a finite dimensional subspace of X, then $ref_{ab}(\mathcal{S}) = \mathcal{S} + ref_{ab}(\mathcal{S}_F)$. In particular, if $\mathcal{S}$ is a finite dimensional linear subspace of $B(X)$, then $ref_{ab}(\mathcal{S}) = \mathcal{S} + ref_{ab}(\mathcal{S}_F)$.*

Proof. By Lemma 3.1[67], for any subspace E of X with a denumerable Hamel basis there exists a separating vector $y \in X$ for $\mathcal{S}_I$ such that $\mathcal{S}_I y \cap E = 0$. Since $\mathcal{S}_F y \subseteq \mathcal{S}_F X$

and $\mathcal{S}_F X$ is finite dimensional, the conclusion follows immediately from Theorem 1.25. □

The next example indicates that Corollary 1.26 is not true for algebraic reflexivity.

Example 1.27. Let $\{e_i\}_{i=1}^{\infty}$ be an orthonormal basis for a Hilbert space H, $\mathcal{S}_F$ be the span of $\{e_1 \otimes e_j : j = 2, 3, ...\}$ and $\mathcal{S} = \mathbf{C}I + \mathcal{S}_F$. Clearly, $\mathcal{S}$ satisfies the conditions of Corollary 1.26, so the conclusion of the corollary follows. However, $ref_a(\mathcal{S}) \neq \mathcal{S} + ref_a(\mathcal{S})$. To see this, we show that $e_1 \otimes e_1 \notin \mathcal{S} + ref_a(\mathcal{S}_F)$ but $e_1 \otimes e_1 \in ref_a(\mathcal{S})$. It follows from Theorem 3.5[67] that $\mathcal{S}_F \subseteq ref_a(\mathcal{S}_F) \subseteq \mathcal{S}_F + F(H) = F(H)$, so $\mathcal{S} + ref_a(\mathcal{S}_F) = \mathbf{C}I + ref_a(\mathcal{S}_F)$. For any $S \in ref_a(\mathcal{S}_F)$, we have $Se_1 \in \mathcal{S}_F e_1 = 0$. Since $e_1 \otimes e_1(e_1) = e_1$, $e_1 \otimes e_1 \notin ref_a(\mathcal{S}_F)$. Thus $e_1 \otimes e_1 \notin \mathcal{S} + ref_a(\mathcal{S}_F)$. Next, we show $e_1 \otimes e_1 \in ref_a(\mathcal{S})$. For any $x \in H$, if $x = \lambda e_1$, then $e_1 \otimes e_1(x) = \lambda e_1 = Ix$. If $(x, e_j) = \alpha \neq 0$ for some $j > 1$, let $(x, e_1) = \beta$. Then $e_1 \otimes e_1(x) = \beta e_1 = \frac{\beta}{\alpha} e_1 \otimes e_j(x)$, so $e_1 \otimes e_1 \in ref_a(\mathcal{S})$. □

Corollary 1.28. *Let $\mathcal{S}$ and $\mathcal{S}_F$ be as in Theorem 1.25. If $\{x : \mathcal{S}_F x = 0\}$ has finite codimension in X, then $ref_{ab}(\mathcal{S}) = \mathcal{S} + ref_{ab}(\mathcal{S}_F)$.*

Proof. Suppose that $\{x \in X : \mathcal{S}_F x = 0\}$ has finite codimension in X. Lemma 3.1[67] implies that for any subspace E of X with a denumerable Hamel basis, there exists a separating vector $y \in \{x \in X : \mathcal{S}_F x = 0\}$ for $\mathcal{S}_I$ such that $\mathcal{S}_I y \cap E = 0$. Clearly, for any $S \in \mathcal{S}_F$, $Sy = 0$, so $\mathcal{S}_F y$ is finite dimensional. The conclusion now follows from Theorem 1.25. □

Next, we prove a technical lemma, which we need to prove Theorem 1.25.

Lemma 1.29. *Let X be a finite dimensional Banach space and $\{u_n\}_{i=1}^{\infty}$ be a sequence in X such that $\lim_{n\to\infty} u_n = u$. Then there exists a positive integer d with $d \leq dim(X)$ such that for any $\epsilon > 0$, there exist positive integers $n_1 <, ..., < n_d$ and scalars $t_1, ..., t_d$ with*

$|t_1| + ... + |t_d| < 1 + \epsilon$ *such that* $u = \sum_{i=1}^{d} t_i u_{n_i}$.

Proof. If $u = 0$, we can take any positive integer $d \leq dimX$ and any positive integers $n_1 <, ..., < n_d$ and $t_1 = ... = t_d = 0$. Suppose $u \neq 0$. Let $E_n = span\{u_n, u_{n+1}, ...\}$ and $d_n = dimE_n$. It follows that $\{d_n\}$ is a non-increasing sequence of positive integers. Let $d = \lim_{n\to\infty} d_n$. Then $1 \leq d \leq dimX$. It is easy to see that $u \in E_n$ for each n and there exists an N such that $d_n = d$ for all $n \geq N$.

If $d = 1$, then $d_n = 1$ and $E_n = \mathbf{C}u$ for all $n \geq N$. This implies that $u_n = a_n u$ for some scalar a_n as $n \geq N$. Since $\lim_{n\to\infty} u_n = u$, we have $\lim_{n\to\infty} a_n = 1$. For any $\epsilon > 0$, choose n_0 large enough so that $0 < \frac{1}{|a_{n_0}|} < 1 + \epsilon$. Now, we can write $u = \frac{1}{a_{n_0}} u_{n_0}$.

If $d > 1$, choose positive integers $n_1, ..., n_{d-1}$ so that $N \leq n_1, ..., n_{d-1}$ and $\{u_{n_1}, ..., u_{n_{d-1}}, u\}$ form a basis of E_N. For any $u_n \in E_N$, write

$$u_n = \sum_{i=1}^{d-1} a_i^{(n)} u_{n_i} + a^{(n)} u. \tag{1.8}$$

It is not hard to show that $\lim_{n\to\infty} u_n = u$ implies that $\lim_{n\to\infty} a_i^{(n)} = 0$ for $i = 1, ..., d-1$ and $\lim_{n\to\infty} a^{(n)} = 1$. Solving for u from (1.8), we obtain $u = \frac{1}{a^{(n)}} u_n - \sum_{i=1}^{d-1} \frac{a_i^{(n)}}{a^{(n)}} u_{n_i}$. For any $\epsilon > 0$, choose n_d large enough so that $|\frac{1}{a^{(n_d)}}| + \sum_{i=1}^{d-1} |\frac{a_i^{(n)}}{a^{(n_d)}}| < 1 + \epsilon$. Let $t_i = -\frac{a_i^{(n_d)}}{a^{(n_d)}}$ for$\frac{d)}{d)}$ for $i = 1, ..., d-1$ and $t_d = \frac{1}{a^{(n_d)}}$. Then we have $u = \sum_{i=1}^{d} t_i u_{n_i}$ with $|t_1| + ... + |t_d| < 1 + \epsilon$. □

Proof of Theorem 1.25.

Clearly, $ref_{ab}(\mathcal{S}) \supseteq \mathcal{S} + ref_{ab}(\mathcal{S}_F)$. We only need to prove the other direction. By Theorem 3.5[67], $ref_a(\mathcal{S}) \subseteq \mathcal{S} + F(X)$. Therefore, $ref_{ab}(\mathcal{S}) \subseteq \mathcal{S} + F(X)$. Thus, we only need to show that $ref_{ab}(\mathcal{S}) \cap F(X) \subseteq ref_{ab}(\mathcal{S}_F)$. For any $T \in ref_{ab}(\mathcal{S}) \cap F(X)$, let $Ran(T)$ denote the range of T. For any $z \in X$, we define $E_z = span\{Ran(T), Az, Ran(B) : A \in \mathcal{S}, B \in \mathcal{S}_F\}$. Then E_z has a denumerable Hamel basis. By our assumption, there exists a separating vector $y \in X$ for $\mathcal{S}_I$ so that $\mathcal{S}_I y \cap E_z = 0$ and $dim(\mathcal{S}_F y) < \infty$. For a fixed vector y and for each positive integer n, $\frac{1}{n}y$ has the same property as y. Since $T \in ref_{ab}(\mathcal{S})$, there exists a fixed $M_T > 0$, such that $Tx \in \mathcal{S}_{M_T}x$ for all $x \in X$. In particular, for any positive integer n, we have $T(z + \frac{1}{n}y) \in \mathcal{S}_{M_T}(z + \frac{1}{n}y)$. Let $A_n \in \mathcal{S}_{M_T}$ such that

$$T(z + \frac{1}{n}y) = A_n(z + \frac{1}{n}y). \tag{1.9}$$

Suppose that $A_n = B_n + C_n$ with $B_n \in \mathcal{S}_I$ and $C_n \in \mathcal{S}_F$. Replacing A_n by $B_n + C_n$ in equation (1.9) and solving for $\frac{1}{n}B_n y$, we get $\frac{1}{n}B_n y = T(z + \frac{1}{n}y) - (B_n + C_n)z - C_n(\frac{1}{n}y)$. This implies that $B_n y \in E_z$. Since $\mathcal{S}_I y \cap E_z = 0$, $B_n y = 0$. This implies $B_n = 0$, since y is a separating vector of $\mathcal{S}_I$. Therefore, equation (1.9) can be reduced to

$$T(z + \frac{1}{n}y) = C_n(z + \frac{1}{n}y). \tag{1.10}$$

Solving for $C_n z$ from (1.10), we obtain

$$C_n z = T(z + \frac{1}{n}y) - \frac{1}{n}C_n y. \tag{1.11}$$

Since $||C_n|| = ||A_n|| \leq M_T$, it follows that $\lim_{n\to\infty} C_n z = Tz$. Let $\tilde{X} = span\{Ran(T), \mathcal{S}_F y\}$. Since $T \in F(X)$ and $dim(\mathcal{S}_F y) < \infty$, we have $dim\tilde{X} < \infty$. Equation (1.11) implies that the sequence $C_n z$ is in $\tilde{X}$. By Lemma 1.29, there exists $d \leq dim\tilde{X}$ such that for $\epsilon = 1$ there exist positive integers $n_1 <, ..., < n_d$ and scalars $t_1, ..., t_d$ with $|t_1| + ... + |t_d| < 2$ with $Tz = \sum_{i=1}^{d} t_i C_{n_i} z$. Since $\sum_{i=1}^{d} t_i C_{n_i} \in \mathcal{S}_F$ and $|| \sum_{i=1}^{d} t_i C_{n_i}|| \leq \sum_{i=1}^{d} |t_i| \, ||C_{n_i}|| \leq \sum_{i=1}^{d} |t_i| M_T < 2M_T$, we have $Tz \in (\mathcal{S}_F)_{2M_T} z$. Since z is arbitrary, we obtain that $T \in ref_{ab}(\mathcal{S}_F)$. □

Theorem 1.30. *Let $\mathcal{S}$ be a subspace of $B(X)$ with a separating vector x. Suppose that M is an invariant vector space of $\mathcal{S}$ containing x and M has an invariant complement N*

in X. Let $y \in N$ and let $\tilde{S} = \{A \in S : Ay = 0\}$. If $\tilde{S}|_M$ is algebraically boundedly reflexive, then S is algebraically boundedly reflexive.

Proof. Let $T \in ref_{ab}(S)$. Then there exists an $M_T > 0$ such that for any $v \in X$, $Tv \in S_{M_T}v$. Thus there exists $A \in S_{M_T}$ such that $Tx = Ax$. For any $y \in N$, choose $\tilde{A} \in S_{M_T}$ with $\tilde{A}(x+y) = T(x+y)$, and $A_y \in S_{M_T}$ with $Ty = A_y y$. Then $T(x+y) = Tx + Ty = Ax + A_y y$, and $\tilde{A}(x+y) = \tilde{A}x + \tilde{A}y$. Thus we have $(A - \tilde{A})x = (A_y - \tilde{A})y$. Since $M \cap N = 0$, we have $(A - \tilde{A})x = 0$. Since x is a separating vector of S, $A = \tilde{A}$ and $A_y y = Ay$. Thus $Ty = Ay$. Let $\tilde{T} = T - A$. Hence we may assume that $TN = 0$. To prove that $T \in S$, it suffices to prove that $TM = 0$. For any $u \in M$, there exist A_u, $A_{u+y} \in S_{M_T}$ such that

$$Tu = A_u u,\ T(u+y) = A_{u+y}(u+y),\ ||A_u|| \le M_T \text{ and } ||A_{u+y}|| \le M_T. \tag{1.12}$$

By (1.12) and $Ty = 0$, it follows that $Tu = T(u+y) = A_{u+y}u + A_{u+y}y = A_u u$. Hence

$$(A_{u+y} - A_u)u = A_{u+y}y. \tag{1.13}$$

Using $M \cap N = 0$ and (1.13), we have that $A_{u+y}y = 0$ and $A_{u+y} \in \tilde{S}$. By (1.12) and (1.13), it follows that $Tu = A_{u+y}u$. Hence $T|_M \in ref_{ab}(\tilde{S}|_M)$. Since $ref_{ab}(\tilde{S}|_M) = \tilde{S}|_M$, we have that $T|_M \in \tilde{S}|_M$ and $T|_M = B|_M$, for some $B \in \tilde{S}$. Since x in M, $Tx = 0$ and x is a separating vector for S, we have $B = 0$. Hence $T|_M = 0$. □

Corollary 1.31. *Let S and $\tilde{S}$ be as in Theorem 1.30. If $dim(\tilde{S}|_M)$ is finite, then S is algebraically boundedly reflexive.*

Proof. By Theorem 1.30, it suffices to prove that $\tilde{S}|_M$ is algebraically boundedly reflexive. Since bounded reflexivity implies algebraic bounded reflexivity, we only need prove that $\tilde{S}|_M$ is boundedly reflexive. Let $T \in ref_b(\tilde{S}|M)$. Let $\tilde{T}$ be the extension of T to $[M]$, where $[M]$ is the norm closure of M. Since $T \in ref_b(\tilde{S}|_M)$, there exists M_T such that for any $x \in M$, $Tx \in [\tilde{S}_{M_T}x]$. Thus for any $x \in [M]$, $\tilde{T}x \in [\tilde{S}_{M_T}x]$. Since $dim(\tilde{S}|_M)$ is finite, we have that $dim(\tilde{S}|_{[M]})$ is finite. Since $[M]$ contains a separating vector of $\tilde{S}$, by Corollary 1.5, we have $\tilde{T} \in \tilde{S}|_{[M]}$. Hence $T \in \tilde{S}|_M$ and $\tilde{S}|_M$ is boundedly reflexive. □

2.2 Relations between bounded reflexivity and complete positivity

Let $\mathcal{S}$ be any subspace of $B(H)$ and $dim\mathcal{S} = n$. By Theorem 1.1.13, $\mathcal{S}$ is $[\sqrt{2n}\,]$-reflexive, where $[t]$ denotes the largest integer that is less than or equal to t. By Example 1.1.14, it follows that for any $n \geq 2$, there exists a subspace $\mathcal{S}$ of $M_m(\mathbf{C})$ with $dim\mathcal{S} = n$ such that $\mathcal{S}$ is not $([\sqrt{2n}\,] - 1)$-reflexive.

The situation is different for bounded reflexivity. In the case of bounded reflexivity, we can prove the following:

Thoerem 2.1. *Let $\mathcal{S}$ be a subspace of $B(H)$ with $dim\mathcal{S} = n$. Then $\mathcal{S}$ is boundedly $[\sqrt{n+1}\,]$-reflexive, where $[t]$ denotes the largest integer that is less than or equal to t.*

To prove Theorem 2.1, we first prove a lemma.

Lemma 2.2. *If $\mathcal{S}$ is a subspace of $M_{k+1}(\mathbf{C})$ and $dim\mathcal{S} \leq (k+1)^2 - 2$, then $\mathcal{S}^{(k)}$ is boundedly reflexive.*

Proof. Since $dim(\mathcal{S}_\perp) \geq 2$, there exists A in $\mathcal{S}_\perp$ such that $rankA \leq k$ and $A \neq 0$. By Theorem 1.12, $\mathcal{M} = \{B \in M_{k+1}(\mathbf{C}) : tr(AB) = 0\}$ is boundedly k-reflexive. By Proposition 1.16, $\mathcal{M}$ has property W^*P_k. Since $\mathcal{S} \subseteq \mathcal{M}$, $\mathcal{S}$ is boundedly k-reflexive by Proposition 1.15. □

Proof of Theorem 2.1.

By Corollary 1.26, we can assume that $\mathcal{S}$ consists of finite rank operators. Since $dim\mathcal{S}$ is finite, we can assume that $\mathcal{S}$ is a subset of $M_n(\mathbf{C})$ for some n and $H = \mathbf{C}^{(n)}$ with the standard orthonormal basis $\{e_i\}_{i=1}^n$. To prove the conclusion of the theorem, we only need to show that if $m \leq (k+1)^2 - 2$ then $\mathcal{S}$ is boundedly k-reflexive. We may also assume that $k \leq n-1$, since the result is obvious for other k. Take $T \in M_n(\mathbf{C})$ so that $T^{(k)} \in ref_b(\mathcal{S}^{(k)})$. We will prove $T^{(k+1)} \in ref_b(\mathcal{S}^{(k+1)})$. Suppose this is not true. Then there exists an $x_0 \in H^{(k+1)}$ such that

$$T^{(k+1)}x_0 \notin [\mathcal{S}_1^{(k+1)}x_0] = \mathcal{S}_1^{(k+1)}x_0.$$

Since $[\mathcal{S}_1^{(k+1)}x_0]$ is a closed convex set, it follows that there exist $y_0 \in H^{(k+1)}$ and real numbers a, b such that for any $A \in \mathcal{S}_1$,

$$Re(A^{(k+1)}x_0, y_0) \leq a < b \leq Re(T^{(k+1)}x_0, y_0). \tag{2.1}$$

Let $x_0 = (x_1, ..., x_{k+1})^t$, $y_0 = (y_1, ..., y_{k+1})^t$ and $B = \sum_{i=1}^{k+1} x_i \otimes y_i$. By (2.1), it follows that for any $A \in \mathcal{S}_1$,

$$Re(tr(AB)) \leq a < b \leq Re(tr(TB)). \tag{2.2}$$

Choose invertible matrices $U, V \in M_n(\mathbf{C})$, such that

$$UBV = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix},$$

where I_r is $r \times r$ identity matrix with $r \leq k+1$.

Let $e = (e_1, ..., e_r)^t$. By (2.1) and (2.2), it follows that for any $A \in \mathcal{S}_1$,

$$Re(P^*V^{-1}AU^{-1}e, e) \leq a < b \leq Re(P^*V^{-1}TU^{-1}e, e), \tag{2.3}$$

where P is the orthogonal projection from $\mathbf{C}^n$ onto $\mathbf{C}^r \oplus 0$. If $r \leq k$, then (2.3) would contradict the fact $T^{(k)} \in ref_b(\mathcal{S}^{(k)})$. Therefore we have $r = k+1$. Since $dim(PV^{-1}\mathcal{S}U^{-1}P)$ $\leq m \leq (k+1)^2 - 2$, we can consider $PV^{-1}\mathcal{S}U^{-1}P$ as a subspace of $M_{k+1}(\mathbf{C})$. By Lemma

2.2, $PV^{-1}SU^{-1}P$ is boundedly k-reflexive. By $T^{(k)} \in ref_b(S^{(k)})$ we have $(PV^{-1}TU^{-1}P)^{(k)} \in ref_b((PV^{-1}SU^{-1}P)^{(k)})$. Since $PV^{-1}SU^{-1}P$ is boundedly k-reflexive, it follows that $PV^{-1}TU^{-1}P \in PV^{-1}SU^{-1}P$. This contradicts (2.3). Therefore $\mathcal{S}$ is boundedly k-reflexive. □

The following example shows that $[\sqrt{n+1}]$ is the smallest integer such that all n-dimensional subspaces of $B(H)$ are boundedly $[\sqrt{n+1}]$-reflexive.

Example 2.3. For any $n \geq 3$, there exist m and a subspace $\mathcal{S}$ of $M_m(\mathbf{C})$ with $dim\, \mathcal{S} = n$ such that $\mathcal{S}$ is not boundedly $([\sqrt{n+1}]-1)$-reflexive.

Proof. For any $n \geq 3$, choose a positive integer l such that $l^2 - 1 \leq n < (l+1)^2 - 1$. Let $k = n - (l^2 - 1)$, $\mathcal{M} = \{T \in M_l(\mathbf{C}) : tr(T) = 0\}$, $\mathcal{A}_k = \{diag(a_1, ..., a_k) : a_i \in \mathbf{C},\ i = 1, ..., k\}$ and $\mathcal{S} = \mathcal{M} \oplus \mathcal{A}_k \subseteq M_{l+k}(\mathbf{C}) = M_m(\mathbf{C})$, where $m = l + k$. It follows from Proposition 1.1 and Theorem 1.12 that $\mathcal{S}$ is boundedly l-reflexive, but $\mathcal{S}$ is not boundedly $(l-1)$-reflexive, where $l = [\sqrt{n+1}\,]$. □

Next, we consider the relations of bounded reflexivity and complete positivity of elementary operators on $B(H)$. One of the motivations for this paper is the following result.

Proposition 2.4[50]. *Let $A_1, ..., A_n$ and T be operators in $B(H)$ and $span\{A_1, ..., A_n\} = \mathcal{S}$. Then the following are equivalent.*

(1) $A_1PA_1^* + ... + A_nPA_n^* \geq TPT^*$ for every positive operator $P \in B(H)$,

(2) For every $x \in H$, there are complex numbers $a_1(x), ..., a_n(x)$ with $|a_1(x)|^2 + ... + |a_n(x)|^2 \leq 1$ such that $Tx = a_1(x)A_1x + ... + a_n(x)A_nx$.

Using the concept of bounded reflexivity, we write the above equivalent conditions as follows:

(1)′ There exists a positive scalar t such that $A_1PA_1^* + ... + A_nPA_n^* \geq tTPT^*$ for every positive operator $P \in B(H)$.

(2)′ $T \in ref_b(\mathcal{S})$.

If we define $\Phi(x) = \sum_{i=1}^{n} A_ixA_i^* - tTxT^*$ for any operator x in $B(H)$, then (1)′ is equivalent to saying that the elementary operator Φ is positive. From (1)′ and (2)′, we know that there exists a positive scalar t such that Φ is positive if and only if $T \in ref_b(\mathcal{S})$. This is exactly the technique used in [76]. Without appealing to the idea of bounded reflexivity, only sufficient conditions are obtained for complete positivity of elementary operators in [76] using the theory of reflexivity of operator spaces. From the above, one can see that it is the bounded reflexivity that describes the positivity of elementary operators.

Corollary 2.5. *Let $\Phi(\cdot) = \sum_{i=1}^{n} A_i(\cdot)B_i$ be an elementary operator on $B(H)$ and let $\mathcal{S} = span\{A_1, ..., A_n\}$. Suppose that every proper subspace of $\mathcal{S}$ is boundedly k-reflexive. Then Φ is completely positive if and only if Φ is k-positive.*

Remark Corollary 2.5 improves Theorem 1[71].

The following corollary is the main result of [104]. Applying Theorem 2.1 and the technique used in Theorem 6[76], we can give a shorter proof of it.

Corollary 2.6. *If $\mathcal{A}$ is a C^*-algebra, $A_i, B_i \in \mathcal{A}$ and $\Phi(\cdot) = \sum_{i=1}^{n} A_i(\cdot)B_i$ is an elementary operator on $\mathcal{A}$, then Φ is completely positive if and only if Φ is $[\sqrt{n}\,]$-positive, where $[t]$ denotes the largest integer that is less than or equal to t.*

In the following, we give another application of Theorem 1.12.

Corollary 2.7. *For any $1 \leq k \leq n-1$, there exists an elementary operator Φ on $M_n(\mathbf{C})$ such that Φ is k-positive and Φ is not $(k+1)$-positive.*

Proof. Choose $T \in M_n(\mathbf{C})$ such that $rankT = k+1$. Let $\mathcal{S} = \{A \in M_n(\mathbf{C}) : tr(AT) = 0\}$. By Theorem 1.12, $\mathcal{S}$ is boundedly $(k+1)$-reflexive, but $\mathcal{S}$ is not boundedly k-reflexive. Suppose $\mathcal{S} = span\{A_1, ..., A_{n^2-1}\}$. Since $dim\mathcal{S} = n^2 - 1$ and $ref_b(\mathcal{S}^{(k)}) \neq \mathcal{S}^{(k)}$, it follows that $ref_b(\mathcal{S}^{(k)}) = M_n(\mathbf{C})^{(k)}$. Hence $T^{(k)} \in ref_b(\mathcal{S}^{(k)})$. By (1)′ and (2)′, there exists a $t > 0$ such that

$$\Phi(x) = \sum_{i=1}^{n^2-1} A_i x A_i^* - tTxT^*$$

is k-positive. Since $\mathcal{S}$ is boundedly $(k+1)$-reflexive, it follows that Φ is not $(k+1)$-positive.

□

Corollary 2.7 gives another proof of Theorem 1[16].

2.3 Applications

In this section, $\mathcal{A}$ denotes a C^*-algebra. $\hat{\mathcal{A}}$ denotes the space of all equivalence classes of irreducible representations of a C^*-algebra $\mathcal{A}$, topologized so that the closure of a subset W of $\hat{\mathcal{A}}$ is the set all those R in $\hat{\mathcal{A}}$ such that $\bigcap_{S \in W} ker(S) \subseteq ker(R)$. A positive element x in $\mathcal{A}$ is called *abelia'n* if the norm closure of $x\mathcal{A}x$ is commutative. If $\mathcal{J}$ is an ideal in a C^*-algebra $\mathcal{A}$, then we say that $\mathcal{J}$ is *essential* in $\mathcal{A}$ if each non-zero closed ideal of $\mathcal{A}$ has a non-zero intersection with $\mathcal{J}$. We say that $\mathcal{A}$ is *antiliminary* if it contains no non-zero abelian elements. $\mathcal{A}$ is said to have a *continuous trace* if it is a liminary C^*-algebra, $\hat{\mathcal{A}}$ is Hausdorff, and if, for each T in $\mathcal{A}$, there is an A in $\mathcal{A}$ and a neighbourhood U of T such that for all π in U, $\pi(A)$ is a one dimensional projection in H_π. An *elementary operator* on $\mathcal{A}$ is a mapping of the form $S : x \to \sum_{i=1}^{n} a_i x b_i$, where a_i and b_i are fixed elements of $\mathcal{A}$. A linear map Φ on $\mathcal{A}$ is *positive* if $\Phi(T)$ is positive for any positive element T in $\mathcal{A}$. We define $\Phi_n = \Phi \otimes I_n : M_n(\mathcal{A}) \to M_n(\mathcal{A})$ by $\Phi \otimes I_n((T_{ij})_{n\times n}) = (\Phi(T_{ij}))_{n\times n}$. Φ is said to be *n-positive* if $\Phi \otimes I_n$ is positive. If Φ is n-positive for all n, then Φ is said to be *completely positive*. Let $||\Phi||_{cb} = sup\{||\Phi_n|| : n \geq 1\}$.

A C^*-algebra $\mathcal{A}$ is an *extension* of a C^*-algebra $\mathcal{B}$ by a C^*-algebra $\mathcal{C}$ if there is a short exact sequence

$$0 \longrightarrow \mathcal{B} \longrightarrow \mathcal{A} \longrightarrow \mathcal{C} \longrightarrow 0.$$

A C^*-algebra $\mathcal{A}$ is said to be *subhomogeneous* with bounded degree n if every irreducible representation of $\mathcal{A}$ is finite dimensional with dimension not greater than n, or equivalently if $\mathcal{A}$ is a C^*-subalgebra of $M_n(\mathcal{B})$ for some commutative C^*-algebra $\mathcal{B}$. We say that a C^*-

algebra $\mathcal{A}$ is *antiliminal by a subhomogeneous C^*-algebra with bounded n* if there exists an exact sequence

$$0 \longrightarrow \mathcal{J} \longrightarrow \mathcal{A} \longrightarrow \mathcal{B} \longrightarrow 0$$

satisfying that $\mathcal{J}$ is subhomogeneous with bounded degree n and $\mathcal{B}$ is an antiliminal C^*-algebra.

Let H denote a complex Hilbert space and let $B(H)$ denote the set of all bounded operators on H, $K(H)$ the set of all compact operators on H. Let $H^{(n)}$ denote the direct sum of n copies of H.

In [74], we prove that if $\mathcal{A}$ is separable and antiliminal, then every positive elementary operator on $\mathcal{A}$ is completely positive. In [3], Archbold, Mathieu and Somerset establish some equivalent conditions on $\mathcal{A}$ which imply that every positive elementary operator on $\mathcal{A}$ is completely positive. In this section, we generalize Theorem 6[3], the main result of [3].

By Theorem 1.2[15], we know that every n-positive linear map of $\mathcal{A}$ into itself is completely positive if and only if $\mathcal{A}$ is subhomogeneous with bounded degree n. For elementary operators on $\mathcal{A}$, we will find that the situation is very different (see Theorem 3.7).

Let

$$0 \to \mathcal{J} \to \mathcal{A} \to \mathcal{B} \to 0$$

be an extension of $\mathcal{B}$ by $\mathcal{J}$. Let $S(x) = \sum_{i=1}^{n} a_i x b_i$ be an elementary operator on $\mathcal{A}$. In this paper, we use the notation of [3]. We denote the ' restriction ' of S to an elementary operator on $\mathcal{J}$ by $S_{|\mathcal{J}}$ and the induced elementary operator on $\mathcal{B}$ by $S_{/\mathcal{B}}$.

Tensoring the above extension by M_n

$$0 \to M_n \otimes \mathcal{J} \to M_n \otimes \mathcal{A} \to M_n \otimes \mathcal{B} \to 0$$

we may canonically identify $(S_n)_{|M_n \otimes \mathcal{J}}$ with $(S_{|\mathcal{J}})_n$ and $(S_n)_{/M_n \otimes \mathcal{B}}$ with $(S_{/\mathcal{B}})_n$.

Lemma 3.1[3]. *Let $\mathcal{A}$ be an extension of $\mathcal{B}$ by $\mathcal{J}$. Then for every $n \geq 1$,*

(1) S is n-positive if and only if $S_{|\mathcal{J}}$ and $S_{/\mathcal{B}}$ are n-positive.

(2) $||S_n|| = max\{||(S_{|\mathcal{J}})_n||, ||(S_{/\mathcal{B}})_n||\}$.

Lemma 3.2[3]. *The following conditions on a C^*-algebra $\mathcal{A}$ are equivalent:*

(1) Every positive elementary operator on $\mathcal{A}$ is completely positive.

(2) For every elementary operator S on $\mathcal{A}$, $||S|| = ||S||_{cb}$.

Lemma 3.3[3]. *Let $\mathcal{A}$ be an antiliminal C^*-algebra. Then there is a dense subset W of $\hat{\mathcal{A}}$ such that $\pi(\mathcal{A})$ is antiliminal for all π in W.*

Lemma 3.4[91]. *Each C^*-algebra $\mathcal{A}$ has a largest postliminal ideal $\mathcal{I}$ and $\mathcal{A}/\mathcal{I}$ is antiliminary.*

Lemma 3.5[91]. *Let $\mathcal{A}$ be a postliminal C^*-algebra. Then $\mathcal{A}$ contaims an essential ideal which has continuous trace.*

Lemma 3.6[107]. *Let $\theta(n)$ be the transpose map in $M_n(\mathbf{C})$. Then*

$$||\theta(n)_k|| = \begin{cases} k, & \text{if } k \leq n, \\ n, & \text{if } k > n. \end{cases}$$

Theorem 3.7. *Let $\mathcal{A}$ be a C^*-algebra and $n \geq 1$. The following statements are equivalent.*

(1) Every n-positive elementary operator on $\mathcal{A}$ is $(n+1)$-positive.

(2) Every n-positive elementary operator on $\mathcal{A}$ is completely positive.

(3) For every elementary operator S on $\mathcal{A}$, $||S_n|| = ||S||_{cb}$.

(4) For every elementary operator ϕ on $\mathcal{A}$, $||S_n|| = ||S_{n+1}||$.

(5) There is a dense subset Γ of $\hat{A}$, where $\hat{A}$ is the space of all equivalence classes of irreducible representations of $\mathcal{A}$, such that $A \to \sum_{\lambda \in \Gamma} \oplus \pi_\lambda(A)$ is a faithful representation of $\mathcal{A}$ with for any $\lambda \in \Gamma$, $dimH_{\pi_\lambda} \leq n$ or $dimH_{\pi_\lambda} = \infty$ and $\pi_\lambda(\mathcal{A}) \cap K(H_{\pi_\lambda}) = 0$, where π_λ means that we pick a representation π_λ from the equivalence class λ in $\hat{A}$.

(6) There exists an exact sequence

$$0 \longrightarrow \mathcal{J} \longrightarrow \mathcal{A} \longrightarrow \mathcal{B} \longrightarrow 0$$

satisfying that $\mathcal{J}$ is subhomogeneous with bounded degree n and $\mathcal{B}$ is an antiliminal C^*-algebra.

Proof. (3) $\Rightarrow$ (4). It is obvious.

(5) $\Rightarrow$ (2). Suppose that S is a n-positive elementary operator on $\mathcal{A}$. Let π be any irreducible represention of $\mathcal{A}$ on H and let S_π be the induced elementary operator on $\pi(\mathcal{A})$. Let π_n be the representation of $M_n(\mathcal{A})$ on $H^{(n)}$ defined by $\pi_n((a_{ij})) = (\pi(a_{ij}))_{n \times n}$. The following commutative diagram

$$\begin{array}{ccc} M_n(\mathcal{A}) & \xrightarrow{\quad S_n \quad} & M_n(\mathcal{A}) \\ \Big\downarrow \pi_n & & \Big\downarrow \pi_n \\ \pi_n(M_n(\mathcal{A})) & \xrightarrow[\quad (S_\pi)_n \quad]{} & \pi_n(M_n(\mathcal{A})) \end{array}$$

and Lemma 1(iv)[3] show that to prove (5) $\Rightarrow$ (2), we only need to prove S_π is completely positive.

If $\pi(\mathcal{A})$ is an irreducible representation of $\mathcal{A}$ on H with $dimH \leq n$, then $\pi(\mathcal{A}) = B(H)$. By Theorem 1[71], we have that every n-positive elementary operator on $B(H)$ is completely positive. By Lemma 1(iv)[3], S_π is n-positive. Hence S_π is completely positive. Let π be an irreducible representation of $\mathcal{A}$ on H such that $dim H = \infty$ and $\pi(\mathcal{A}) \cap K(H) = 0$. Since π is an irreducible representation of $\mathcal{A}$ on H, it follows that $\overline{\pi(\mathcal{A})} = B(H)$. Thus we can extend S_π to an elementary operator $\hat{S}_\pi$ on $B(H)$. By S_π is n-positive, it follows that $\hat{S}_\pi$ is n-positive on $B(H)$.

By $\pi(\mathcal{A}) \cap K(H) = 0$, we have

$$\pi(\mathcal{A})/K(H) \cong \pi(\mathcal{A}).$$

Let τ be the canonical map from $B(H)$ into $B(H)/K(H)$. Let $\overline{\Phi}(\tau(A)) = \sum_{i=1}^{n} \tau(a_i)\tau(x)\tau(b_i)$, if $\Phi(x) = \sum_{i=1}^{n} a_i x b_i$ on $B(H)$. Thus by Theorem 4[74], $\overline{\hat{S}_\pi}$ is completely positive on $B(H)/K(H)$. Hence $\hat{S}_\pi$ is completely positive, and S_π is completely positive.

(1) $\Rightarrow$ (6). Suppose that (6) is not true. Let $\mathcal{I}$ the largest postliminal ideal of $\mathcal{A}$. Then $\mathcal{I}$ is not subhomogeneous with bounded degree n.

Let π be an irreducible representation of $\mathcal{I}$ on H such that $dim H \geq n+1$. By Theorem 6.1.5[91], we have that $K(H) \subseteq \pi(\mathcal{I})$. By section 2.2, we can construct an elementary operator S on $\mathcal{I}$ such that S_π is n-positive and S_π is not $(n+1)$-positive on $\pi(\mathcal{I})$. Hence S is not $(n+1)$-positive. Let $S_{|\mathcal{I}}(x) = \sum_{i=1}^{m} a_i x b_i$, $a_i, b_i \in \mathcal{I}$ be the elementary operator we construct. Then S is n-positive and S is not $(n+1)$-positive on $\mathcal{I}$. Let

$$\widetilde{S}(x) = \sum_{i=1}^{m} a_i x b_i, \ x \in \mathcal{A}.$$

By Lemma 3.1, it follows that $\widetilde{S}$ is n-positive and $\widetilde{S}$ is not $(n+1)$-positive. This is a contradiction.

(4) $\Rightarrow$ (6). We denote by $\mathcal{I}$ the largest postliminal ideal of $\mathcal{A}$. By Lemma 5, let J be

an essential ideal of $\mathcal{I}$ with a continuous trace. Suppose that (6) is not true. Since $\mathcal{J}$ is an essential ideal, it follows that $\mathcal{J}$ is not subhomogeneous with bounded degree n. Let π be an irreducible represention of $\mathcal{J}$ with $dim H_\pi \geq n+1$. Since $\mathcal{J}$ has a continuous trace, we have $\pi(\mathcal{J}) \supseteq K(H)$ and $\hat{\mathcal{J}}$ is a locally compact and Hausdorff space. We may assume that $\pi(\mathcal{J}) = M_{n+1}(\mathbf{C})$. Let $\{E_{ij}\}$ be matrix units of M_{n+1}. By [31, 3.1, 3.3, 4.1], there is an open neighbourhood V of π in $\hat{\mathcal{J}}$ and $\{e_{ij}\} \subseteq \mathcal{J}$ such that $\pi(e_{ij}) = E_{ij}$, and $\sigma(e_{ii})$, $(i = 1, ..., n+1)$ are rank-one projections and $\sigma(e_{ij})$, $(i < j)$ are partial isometries with initial projection $\sigma(e_{jj})$ and final projection $\sigma(e_{ii})$. for all $\sigma \in V$. Let $e_{ji} = e_{ij}^*$ for $j < i$.

Since $\hat{\mathcal{J}}$ is a locally compact Hausdorff space, there is a continuous function $g : \hat{\mathcal{J}} \to [0, 1]$ supported in V. Let θ be the transpose map of $M_{n+1}(\mathbf{C})$. Let $\theta(x) = \sum_{i=1}^{t} \alpha_i U_i x V_i$ on $M_{n+1}(\mathbf{C})$, where $U_i, V_i \in \{E_{ij}\}$, $\alpha_i \in C$. For $x \in \mathcal{A}$, let $S(x) = \sum_{i=1}^{t} \alpha_i a_i x b_i$, where $a_i, b_i \in \{e_{ij}\}$, $\pi(a_i) = U_i$, $\pi(b_i) = V_i$. For $\sigma \in \hat{\mathcal{J}}$ define $S_\sigma(y) = (\sum_{i=1}^{t} \alpha_i \sigma(a_i) y \sigma(b_i)) g(\sigma)$, on $\sigma(\mathcal{J})$. Then $S_\sigma = 0$, for any $\sigma \in \hat{\mathcal{J}} \backslash V$. For $\sigma \in V$, by Lemma 3.6, $||(S_\sigma)_k|| = g(\sigma)k$, if $k \leq n+1$ and $||(S_\sigma)_k|| = g(\sigma)(n+1)$, if $k > n+1$. Hence $||(S_{|\mathcal{J}})_n|| = n$, $||(S_{|\mathcal{J}})_{n+1}|| = n+1$. Since $S_{/\mathcal{B}} = 0$, where $\mathcal{B} = \mathcal{A}/\mathcal{J}$, by Lemma 1, $||S_n|| \neq ||S_{n+1}||$. This is a contradiction to (4).

(6) $\Rightarrow$ (5). Let $W = \{\sigma \in \hat{\mathcal{B}} : \ \sigma(\mathcal{B}) \cap K(H_\sigma) = 0\}$. By the proof Lemma 3[3], we have that W is dense in $\hat{\mathcal{B}}$. Let τ be an isomorphism from $\mathcal{A}/\mathcal{J}$ onto $\mathcal{B}$. For any $\sigma \in \hat{\mathcal{B}}$, define $\pi_\sigma(x) = \sigma(\tau(x))$ for x in $\mathcal{A}$. Then π_σ is an irreducible representation of $\mathcal{A}$. Let $\Gamma = \{\pi \in \hat{\mathcal{A}} : \pi|_{\mathcal{J}} \neq 0\} \cup \{\pi_\sigma : \sigma \in W\}$. Let x in $\mathcal{A}$. Suppose that $x \in \mathcal{J}$. Then there exists $\pi \in \Gamma$ such that $\pi(x) \neq 0$. Suppose $x \neq \mathcal{J}$. Since $\hat{W}$ is dense, it follows that there exists a

σ in $\hat{W}$ such that $\sigma(\tau(x)) \neq 0$. Thus $\pi_\sigma(x) \neq 0$. Hence Γ is dense in $\hat{\mathcal{A}}$.

(5) $\Rightarrow$ (3). Let S be an elementary operator on $\mathcal{A}$. By Lemma 1(iv)[3], it follws that for $n \geq 1$,

$$||S_n|| = sup\{||(S_\lambda)_n|| : \lambda \in \Gamma\}.$$

For $\lambda \in \Gamma$, by Proposition 7.9[93], then $||(S_\lambda)_n|| = ||S_\lambda||_{cb}$. For $\lambda \in \Gamma$, if $\pi_\lambda(\mathcal{A}) \cap K(H_\lambda) = 0$, then $||S_\lambda|| = ||S_\lambda||_{cb}$.

(2) $\Rightarrow$ (1). It is obvious. □

By Lemma 3.6 and Theroem 3.7, it is easy to show

Corollary 3.8[16]. *$P_{n-1}(M_n(\mathbf{C}), M_n(\mathbf{C})) \supset P_n(M_n(\mathbf{C}), M_n(\mathbf{C}))$ where $P_i(M_n(\mathbf{C}), M_n(\mathbf{C}))$ is the set of all i-positive maps from $M_n(\mathbf{C})$ into itself.*

Remark Let $\mathcal{A}$ be a prime C^*-algebra and let $\mathcal{A} \otimes_h \mathcal{A}$ denote the Haagerup tensor product of $\mathcal{A}$ with itself. Define $\theta : \mathcal{A} \otimes_h \mathcal{A} \to CB(\mathcal{A})$, (where $CB(\mathcal{A})$ is the algebra of completely bounded operators on $\mathcal{A}$ with the completely bounded norm $||\cdot||_{cb}$) by

$$\theta(\sum_{i=1}^{n} a_i \otimes b_i)(c) = S(c)$$

for $c \in \mathcal{A}$, where $S(c) = \sum_{i=1}^{n} a_i c b_i$. By Corollary 3.9[2], θ is an isometry. Hence by Theorem 7, if $\mathcal{A}$ is prime and antiliminal by a subhomogeneous C^*-algebra with bounded n, then $||\sum_{i=1}^{n} a_i \otimes b_i||_h = ||S_n||$.

Chapter 3

Derivations and Cohomology

3.1 Derivations

3.1.1 Derivations on certain subalgebras of $B(H)$

In this section, we unify some results on derivations by considering derivations from an algebra $\mathcal{A}$ containing all rank one operators of a nest algebra into an $\mathcal{A}$-bimodule $\mathcal{B}$. Chernoff [15] proves that every derivation from $F(H)$ into $B(H)$ is inner. In [18], Christensen proves that every derivation from a nest algebra into itself or into $B(H)$ is inner. In [18], Christensen and Peligrad show that every derivation of a quasitriangular operator algebra into itself is inner. Knowles [59] generalizes the result of [18] and gets that any derivation from a nest algebra into an ideal $\mathcal{J}$ of $B(H)$ is inner. Let $\mathcal{N}$ be a nest of subspaces of a Hilbert space H, let $\mathcal{A}$ be a subalgebra of $B(H)$ containing all rank one operators of $alg\mathcal{N}$, and let δ be a derivation from $\mathcal{A}$ into $B(H)$. We prove that if one of the following conditions holds:

1. $H_- \neq H$,
2. $0_+ \neq 0$,
3. there exists a nontrivial $P \in \mathcal{N}$, such that $P \in \mathcal{A}$,

then δ is inner.

We also prove that for any nest, if δ is a norm continuous derivation from $\mathcal{A}$ into $B(H)$, then δ is inner. We discuss some applications of these results.

Let $F_1(H)$ be the subset of all operators in $F(H)$ with rank less than or equal to 1. We call a subalgebra $\mathcal{A}$ of $B(H)$ *standard* provided $\mathcal{A}$ contains $F(H)$. For a nest $\mathcal{N}$ on H, $alg\mathcal{N}$ is said to be the *nest algebra* associated with $\mathcal{N}$. If $\mathcal{A}$ is a subalgebra of $B(H)$, then we say that $\mathcal{A}$ is a *triangular* operator algebra, if $\mathcal{A} \cap \mathcal{A}^*$ is a maximal abelian selfadjoint subalgebra of $B(H)$. If $\mathcal{J}$ is maximal triangular, by Lemma 2.3.3[54], it follows that $lat\mathcal{J}$ is a nest. For a maximal triangular algebra $\mathcal{J}$ if $lat\mathcal{A}$ is a maximal nest, we say that $\mathcal{A}$ is *strongly reducible*. A *derivation* δ of an algebra $\mathcal{A}$ into an $\mathcal{A}$-bimodule $\mathcal{B}$ is a linear map satisfying $\delta(AB) = A\delta(B) + \delta(A)B$, for any $A, B \in \mathcal{A}$. In this section, we do not assume that the derivation is bounded. A derivation δ is called $\mathcal{B}$-inner if there exists $T \in \mathcal{B}$, such that $\delta(A) = AT - TA$. When we say that a derivation $\delta : \mathcal{A} \to \mathcal{B}$ is inner, we mean $\mathcal{B}$-inner.

Let $\mathcal{N}$ be a nest. In the following, we consider the derivation from a subalgebra $\mathcal{A}$ of $B(H)$ containing all rank one operators of $alg\mathcal{N}$ into $B(H)$.

Theorem 1.1. *If $\mathcal{N}$ is a nest such that $H_- \neq H$, $\mathcal{A}$ is a subalgebra of $B(H)$ containing $(alg\mathcal{N}) \cap F_1(H)$, and δ is a derivation from $\mathcal{A}$ into $B(H)$, then δ is inner.*

Proof. Since $H_- \neq H$, for any $f \in (H_-)^\perp$, $f \neq 0$, we choose y in H such that $(y, f) = 1$. For any x in H, by Lemma 3.7[96], it follows that $x \otimes f \in alg\mathcal{N}$. Now define

$$Tx = -\delta(x \otimes f)y, \quad \text{for } x \text{ in } H.$$

Now for A in $\mathcal{A}$,

$$TAx = -\delta(Ax \otimes f)y = -\delta(A)x - A\delta(x \otimes f)y = -\delta(A)x + ATx.$$

Hence for any $x \in H, -TAx + ATx = \delta(A)x$; thus

$$\delta(A) = AT - TA.$$

It remains to show that δ is bounded.

Let $\lim_{n\to\infty} x_n = x$, and $\lim_{n\to\infty} Tx_n = y$. Now for any rank one operator $A \in alg\mathcal{N}$, we have that $\delta(A)$ and TA are bounded. It follows that $AT = \delta(A) + TA$ is bounded, and $\lim_{n\to\infty} ATx_n = ATx = Ay$. Since $\mathcal{A}$ contains all rank one operators of $alg\mathcal{N}$, and every finite rank operator of $alg\mathcal{N}$ is a sum of some rank one operators of $alg\mathcal{N}$ (Proposition 3.8[20]), we have, for any finite rank operator B in $alg\mathcal{N}$, $BTx = By$. By Theorem 3.11[20], choose a bounded net $\{B_\lambda\}$ of finite rank operators in $alg\mathcal{N}$ such that $\lim_\lambda B_\lambda = I$ in the strong operator topology. We have $Tx = y$. By the Closed Graph Theorem, it follows that T is bounded. □

Corollary 1.2. *If $\mathcal{N}$ is a nest such that $0_+ \neq 0$, and $\mathcal{A}$ is a subalgebra of $B(H)$ containing all rank one operators of $alg\mathcal{N}$, then every derivation δ from $\mathcal{A}$ into $B(H)$ is inner.*

Proof. Let $\mathcal{N}^\perp = \{N^\perp : N \in \mathcal{N}\}$. Then $\mathcal{N}^\perp$ is a nest such that $H_- \neq H$. Since $alg\mathcal{N}^\perp = (alg\mathcal{N})^*$, it follows that $\mathcal{A}^*$ contains all rank one operators of $alg\mathcal{N}^\perp$. Define $\delta^*(A) = (\delta(A^*))^*$ for any A in $\mathcal{A}^*$. It is easy to prove that δ^* is a derivation from $\mathcal{A}^*$ into $B(H)$. By Theorem 1.1, we have that δ^* is inner. It follows that δ is inner. □

We now consider a nest $\mathcal{N}$ such that $H_- = H$.

Lemma 1.3. *Let $\mathcal{N}$ be a nest, $E_1, E_2 \in \mathcal{N}$ and $E_1 \subset E_2$. If T is a linear map from E_2 into H such that $ST = TS$ on E_2 for any rank one operator S of $alg\mathcal{N}$, then there exists λ such that $Tx = \lambda x$, for any $x \in E_1$.*

Proof. For $x \in E_1$, choose $y \in E_2 - E_1$ such that $||y|| = 1$. Since $x \otimes y \in alg\mathcal{N}$, by hypothesis

$$x \otimes Ty(y) = (x \otimes y)Ty = Tx = (Ty, y)x.$$

Since every one dimensional subspace of $L(E_2, H)$ is reflexive, it follows that there exists λ such that $T = \lambda I$. □

Lemma 1.4. *Let $\mathcal{N}$ be a nest such that $H_- = H$, and let $M = \cup\{N : N \subset H, N \in \mathcal{N}\}$. Then there exists a linear map T from M into H such that $\delta(A)x = (AT - TA)x$, for any x in M.*

Proof. Since $H_- = H$, we may choose an increasing sequence $\{P_i\} \subseteq \mathcal{N}$ such that $P_i \to I$ in the strong operator topology. Also choose $f \in P_i^{\perp}$, and $y \in H$, such that $||f|| = 1, (y, f) = 1$ and $||y|| \leq 2$. Define

$$T_i x = -\delta(x \otimes f)y \tag{1.1}$$

for $x \in P_i$. Using an argument similar to the proof of Theorem 1.1, we may prove that for A in $\mathcal{A}$, $\delta(A)x = (AT_i - T_iA)x$ for x in P_i. If $j \geq i$, then for $x \in P_i, (AT_i - T_iA)x = (AT_j - T_jA)x$. Hence

$$A(T_i - T_j)x = (T_i - T_j)Ax, \text{ for } x \in P_i. \tag{1.2}$$

By Lemma 1.3, we have $T_j - T_i = \lambda_{ij}$ on P_{i-1}. Now for $j > 2$, let $\tilde{T}_j = T_1 + \lambda_{1,j}$. We have, for $k > j > 2$, $\tilde{T}_j x = \tilde{T}_k x$ for all $x \in P_{j-1}$. Now for any $x \in \cup\{P_i\} = \cup\{N : N \subset H, N \in \mathcal{N}\}$, choose a $j > 2$ such that $x \in P_j$ and let $Tx = \tilde{T}_j x$. Then, T is well defined and for x in M, $\delta(A)x = (AT - TA)x$. □

Remark Using the idea in the proof of Theorem 1.1, we can prove that in Lemma 1.4, T_i is a bounded operator from P_i into H.

Theorem 1.5. *If $\mathcal{N}$ is a nest, $\mathcal{A}$ is a subalgebra of $B(H)$ containing all rank one operators of alg$\mathcal{N}$, and δ is a norm continuous derivation from $\mathcal{A}$ into $B(H)$, then δ is inner.*

Proof. If $\mathcal{N}$ satisfies $H_- \neq H$, then by Theorem 1.1, δ is inner. If $\mathcal{N}$ satisfies $H_- = H$, then by Lemma 1.4, there exists a linear map T such that

$$\delta(A)x = (AT - TA)x, \quad \text{for any } x \text{ in } M = \cup\{N : N \subset H, N \in \mathcal{N}\}.$$

By (1.1) and the boundedness of δ, it follows that $||T_i x|| \leq 2||\delta|| ||x||$. Since $|\lambda_{ij}| \leq ||T_i|| + ||T_j|| \leq 4||\delta||$, it follows that $||T|| \leq 6||\delta||$. Thus T is bounded on M. Let $\tilde{T}$ be the unique bounded extension of T to H. Then $\tilde{T}$ is bounded and for A in $\mathcal{A}$, $\delta(A) = A\tilde{T} - \tilde{T}A$. □

Theorem 1.6. *Let $\mathcal{N}$ be a nest satisfying $H_- = H$. If there exists a nontrivial projection $P \in \mathcal{N}$, such that $P \in \mathcal{A}$, and δ is a derivation from $\mathcal{A}$ into $B(H)$, then δ is inner.*

Proof. As in the proof of Lemma 1.4, we choose $P_1 = P$. Let $H = P \oplus P^{\perp}$. Then T can be decomposed as

$$T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}.$$

Let $Q = \cup\{N - P : P \subset N \in \mathcal{N}, N \neq H\}$, $T_{12} : Q \to P$, $T_{22} : Q \to Q$.

By the definition of T, T_{11} and T_{21} are bounded. We now prove that T_{12} and T_{22} are bounded. Since

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in \mathcal{A},$$

we have that

$$\delta(A) = \begin{pmatrix} 0 & T_{12} \\ -T_{21} & 0 \end{pmatrix}$$

holds on M. Since $\delta(A)$ is bounded, it follows that T_{12} is bounded. Now, for any rank one operator $A \in B(H)$, we have $PA(1-P) \in \mathcal{A}$. Hence,

$$\delta(PA(1-P)) = \begin{pmatrix} PA(1-P) & PA(1-P)T_{22} - T_{11} \\ 0 & -T_{21}PA(1-P) \end{pmatrix}$$

holds on M. Since $\delta(PA(1-P))$ is bounded, it follows that $PA(1-P)T_{22}$ is bounded. Hence for any $f \in P^{\perp}$ and $e \in P, e \neq 0$, $eT_{22} \otimes f$ is bounded on Q. Thus there exists c such that $|(T_{22}x, f)| \leq c$, for any $x \in Q$, and $||x|| \leq 1$. By the Uniform Boundedness Theorem, we have that $\{||T_{22}x|| : ||x|| \leq 1\}$ is bounded. Hence T_{22} is bounded. As in Theorem 1.5, there exists a bounded extension $\tilde{T}$ of T to H such that for A in $\mathcal{A}$, $\delta(A) = A\tilde{T} - \tilde{T}A$. □

Now we apply the results above to some special subalgebras of $B(H)$. If $A \supseteq F(H)$, then by Theorem 1.1, we have the following:

Corollary 1.7[15]. *Every derivation from a standard operator algebra into $B(H)$ is inner.*

Corollary 1.8[18]. *If δ is a derivation from $alg\mathcal{N}$ into itself, then δ is inner.*

Proof. By Theorems 1.1 and 1.6, there exists T in $B(H)$ such that for any A in $\mathcal{A}$, $\delta(A) = AT - TA$. Now we prove that T is in $alg\mathcal{N}$. For any P in $\mathcal{N}$, since $\delta(P) = PT - TP$ in $alg\mathcal{N}$, we have that $(I-P)\delta(P)P = 0 = -(I-P)TP$. This completes the proof. □

Let $\mathcal{B}$ be a subalgebra of $B(H)$, and let $\mathcal{S}$ be any subset of $B(H)$. We denote by $C(\mathcal{B}, \mathcal{S})$ the collection $\{T \in B(H) : AT - TA \in \mathcal{S},\ \text{for } A \in \mathcal{B}\}$.

Lemma 1.9[58]. *Let $\mathcal{B}$ be a nest algebra and $\mathcal{J}$ be an ideal in $B(H)$. Then $C(\mathcal{B},\mathcal{J}) = CI + \mathcal{J}$.*

Using this Lemma and Theorem 1.6, we easily prove the following result.

Corollary 1.10. *If $\mathcal{B}$ is an algebra containing $alg\mathcal{N}$, then any derivation $\delta : \mathcal{B} \to C_p$ is inner for $1 \leq p \leq \infty$.*

Corollary 1.11. *If $\mathcal{B}$ is a triangular operator algebra containing every rank one operator in $alg\mathcal{N}$, then every derivation δ from $\mathcal{B}$ into $B(H)$ is inner.*

Proof. Suppose $\tilde{\mathcal{N}}$ is a maximal nest containing $\mathcal{N}$. By hypothesis we have that $B \supseteq (alg\mathcal{N}) \cap F_1(H) \supseteq (alg\tilde{\mathcal{N}}) \cap F_1(H)$. Since $\mathcal{B}$ contains all rank one operators of alg $\mathcal{N}$, we have that $lat\mathcal{B} \subseteq \mathcal{N}$. By Theorem 4[27], it follows that $lat\mathcal{B} = \tilde{\mathcal{N}} = \mathcal{N}$. Since $\mathcal{B}$ is a triangular operator algebra, it follows that $\tilde{\mathcal{N}} \subseteq \mathcal{B}$.

If $H_- \neq H$, then by Theorem 1.1, we have that δ is inner.

If $H_- = H$, $\mathcal{N} \subseteq \mathcal{B}$, and $\mathcal{N}$ is a maximal nest, by Theorem 1.6, it follows that δ is inner.

□

Remark By Corollary 1.7, it follows that every derivation $\delta : F(H) \to B(H)$ is inner. However if $\mathcal{B}$ is a unital algebra containing $F(H)$ and $\mathcal{B} \neq B(H)$, then there is a derivation from $F(H)$ into $\mathcal{B}$ that is not inner, e.g., $\delta = \delta_T$ with $T \notin \mathcal{B}$. Also if $\mathcal{A} = K(H) + \mathbf{C}I$, and $T \notin \mathcal{A}$, then $\delta_T : \mathcal{A} \to \mathcal{A}$ is a derivation that is not inner, but $\mathcal{A}$ contains all rank one operators of $B(H)$.

By Lemma 2.3.3[54], we know that if $\mathcal{B}$ is a strongly reducible maximal triangular algebra, then $lat\mathcal{B}$ is a nest and $\mathcal{B}$ contains all rank one operators of $alglat(\mathcal{B})$. Hence by Theorem 1.6 and Corollary 1.11, we have the following result:

Corollary 1.12. *Every derivation from a strongly reducible maximal triangular algebra into $B(H)$ is inner.*

3.1.2 Derivations on nest-subalgebras of von Neumann algebras

Let $\mathcal{N}$ be a nest on H. An $\mathcal{N}$-interval is a projection $E = M - N$ with $M,\ N \in \mathcal{N}$. A *commutative subspace lattice* is a subspace lattice which consists of mutually commuting projections. A commutative subspace lattice has *finite-width* if it is generated by finitely many nests. We have in descending order of the following classes of lattices: commutative subspace lattices, finite-width lattices, tensor products of nests, nests. If $\mathcal{M}$ is a von Neumann algebra and $\mathcal{L}$ is a subspace lattice in $\mathcal{M}$, we denote $alg_{\mathcal{M}}\mathcal{L} = \mathcal{M} \cap alg\mathcal{L}$ and $\mathcal{D}_{\mathcal{L}} = alg_{\mathcal{M}}\mathcal{L} \cap (alg_{\mathcal{M}}\mathcal{L})^*$. If $\mathcal{N}$ is a nest in $\mathcal{M}$, $alg_{\mathcal{M}}\mathcal{N}$ is called a *nest-subalgebra* of $\mathcal{M}$.

Suppose δ is a derivation from $alg_{\mathcal{M}}\mathcal{N}$ into $\mathcal{M}$ and E is a $\mathcal{N}$-interval. Let $\delta_1(EAE) = E\delta(A)E$ for any $A \in alg_{\mathcal{M}}\mathcal{N}$. Then δ_1 is a derivation from $E(alg_{\mathcal{M}}\mathcal{N})E = alg_{E\mathcal{M}E}E\mathcal{N}$ into $E\mathcal{M}E$. In the following, we study the derivations on nest-subalgebras of factor von Neumann algebras.

In [25], Hongke Du and Jianhua Zhang show that every derivation on a nest-subalgebra of a factor von Neumann algebra is inner. But their proof has some gaps. By using some results in [94], we can only show that if $\mathcal{A}$ is a nest-algebra of a type II_∞ or type III factor, then every derivation from $\mathcal{A}$ into itself is inner. Lance [64] shows that $\mathcal{A}$ is a nest-algebra of a type I factor then every derivation from $\mathcal{A}$ into itself is inner.

Lemma 1.13. *Let $\mathcal{M}$ be a factor von Neumann algebra and let $\mathcal{L}$ be a commutative subspace lattice in $\mathcal{M}$. If δ is a derivation from $\mathcal{D}_{\mathcal{L}}$ into a weakly closed $alg_{\mathcal{M}}\mathcal{L}$-bimodule $\mathcal{B}$ in $\mathcal{M}$ containing $alg_{\mathcal{M}}\mathcal{L}$, then δ is inner.*

Proof. Since $\mathcal{L}''$ is a commutative von Neumann algebra, by Theorem 10.8[20], it follows that there exists $m \in \mathcal{B}$ such that $\delta|_{\mathcal{L}''} = \delta_m$. Let $\overline{\delta} = \delta - \delta_m$. For T in $\mathcal{D}_{\mathcal{L}}$ and P in $\mathcal{L}$, by $TP = PT$, it follows that

$$\overline{\delta}(T)P = \overline{\delta}(TP) = \overline{\delta}(PT) = P\overline{\delta}(T).$$

Hence $\overline{\delta}(T)$ belongs to $\mathcal{D}_{\mathcal{L}}$ and δ is a derivation from $\mathcal{D}_{\mathcal{L}}$ into itself. By Theorem 1[99], we have $\overline{\delta} = \delta_a$ for some $a \in \mathcal{D}_{\mathcal{N}}$. Thus $\delta = \delta_{m+a}$ and $m + a \in \mathcal{B}$. □

Lemma 1.14. *Let $\mathcal{M}$ be a factor von Neumann and let $\mathcal{A}$ be a nest-subalgebra of $\mathcal{M}$. The following are equivalent.*

(1) Every derivation from $\mathcal{A}$ into itself is inner.

(2) Every derivation from $\mathcal{A}$ into every weakly closed $\mathcal{A}$-bimodule in $\mathcal{M}$ containing $\mathcal{A}$ is inner.

(3) Every derivation from $\mathcal{A}$ into every weakly closed $\mathcal{A}$-bimodule in $\mathcal{M}$ containing $\mathcal{A}$ such that $\delta(D_{\mathcal{N}}) = 0$ is inner.

(4) Every derivation from $\mathcal{A}$ into itself such that $\delta(D_{\mathcal{N}}) = 0$ is inner.

Proof. We only prove (1)$\Rightarrow$ (2). The rest are proved using the same method. Let $\mathcal{B}$ be a weakly closed $\mathcal{A}$-bimodule in $\mathcal{M}$ containing $\mathcal{A}$. Since $\mathcal{N}''$ is a commutative von Neumann algebra, by Theorem 10.8[20], there exists an element $m \in \mathcal{B}$ such that $\delta|_{\mathcal{N}''} = \delta_m$. Let $\overline{\delta} = \delta - \delta_m$. Then for any a in $\mathcal{N}''$, $\overline{\delta}(a) = 0$. For any $b \in alg_{\mathcal{M}}\mathcal{N}$, by

$$P^{\perp}\overline{\delta}(b)P = \overline{\delta}(P^{\perp}bP) = \delta(0) = 0$$

it follows that $\overline{\delta}$ is a derivation from $alg_{\mathcal{M}}\mathcal{N}$ into itself. By (1), it follows that $\overline{\delta} = \delta_a$ with a in $alg_{\mathcal{M}}\mathcal{N}$. Hence $\delta = \delta_a + \delta_m = \delta_{a+m}$, $a + m \in \mathcal{B}$. □

Remark In [25], Du and Zhang show that (1) and (4) are equivalent.

Lemma 1.15[25]. *Let $\mathcal{N}_1$ and $\mathcal{N}_2$ be nests in a factor von Neumann algebra $\mathcal{M}$ such that $\mathcal{N}_1 \subseteq \mathcal{N}_2$. If every derivation from $alg_{\mathcal{M}}\mathcal{N}_2$ into $\mathcal{M}$ is inner, then every derivation from $alg_{\mathcal{M}}\mathcal{N}_1$ into $\mathcal{M}$ is inner.*

Lemma 1.16[25]. *Let $\mathcal{N}$ be a nest in an infinite factor von Neumann algebra$\mathcal{M}$ and let δ be a derivation from $alg_{\mathcal{M}}\mathcal{N}$ into $\mathcal{M}$ such that $\delta(D_{\mathcal{N}}) = 0$. If there exists a infinite $\mathcal{N}$-interval E such that δ_1 is inner, where $\delta_1(EAE) = E\delta(A)E$ for $A \in alg_{\mathcal{M}}\mathcal{N}$, then δ is inner.*

Lemma 1.17[94]. *Let $\mathcal{M}$ be a factor, $\mathcal{A}$ be an atomic nest-subalgebra of $\mathcal{M}$ and ϕ be an automorphism of $\mathcal{A}$ such that $\phi(x) = x$ for any $x \in \mathcal{D}_{\mathcal{N}}$. Then $\phi = au(A)$ for an invertible element A in $\mathcal{A}$.*

Lemma 1.18[94]. *Let $\mathcal{A}$ be a nest-subalgebra of a factor von Neumann algebra M. For every T in M,*

$$d(T, \mathbf{C}I) \leq 2||\delta_T|_{\mathcal{A}}||.$$

Corollary 1.19. *Let $\mathcal{A}$ be an atomic nest-subalgebra of a factor von Neumann algebra M and let ϕ be an automorphism of $\mathcal{A}$ such that $\phi(x) = x$ for x in $\mathcal{D}_{\mathcal{N}}$ and $||id - \phi|| < 1/2$. Then there is an invertible element A in $\mathcal{A}$ with $||A - I|| \leq 4||\phi - id||$ such that $\phi = au(A)$*

Proof. By Lemma 1.17, $\phi = au(A)$ with $A \in alg_{\mathcal{M}}\mathcal{N}$. By Lemma 2.6[35], we have that A is unique up to a scalar factor. We can choose $||A|| = 1$. For $T \in alg_{\mathcal{M}}\mathcal{N}$,

$$||\delta_A(T)|| = ||AT - TA|| = ||(\phi(T) - T)A|| \leq ||\phi - id||\ ||T||.$$

By Lemma 1.18 there exists a scalar λ such that $||A - \lambda I|| \leq ||\phi - id||$. Since $||\phi - id|| < 1/2$

and $||A|| = 1$, we have $\lambda \neq 0$. By $||A|| = 1$, it follows that

$$||A - \lambda I|| \geq |1 - |\lambda||.$$

Hence $||A - \lambda/|\lambda| I|| \leq 4||\phi - id||$. Replace A by $\overline{\lambda}/|\lambda| A$. □

Proposition 1.20. *Let $\mathcal{A}$ be an atomic nest-subalgebra of a factor von Neumann algebra M. Suppose that δ is a derivation from $\mathcal{A}$ into itself. Then δ is inner.*

Proof. By Proposition II 1[94], it follows that δ is bounded. Let $\phi_t = e^{t\delta}$, $t \in \mathbf{R}$ be the continuous group of automorphisms. By Lemma 1.14, we can assume that $\delta(\mathcal{D}_{\mathcal{N}}) = 0$. Then $\phi_t(x) = x$, for any $x \in \mathcal{D}_{\mathcal{N}}$ and any $t \in \mathbf{R}$. By Lemma 1.17, it follows that there exists $A_t \in alg_{\mathcal{M}}\mathcal{N}$ such that $\phi_t = au(A_t)$. By Corollary 1.19, we can choose A_t such that

$$||A_t - I|| \leq 4||\phi_t - id||.$$

By Lemma 19.3[20], $t^{-1}||\phi_t - id||$ is bounded. Let $D_n = n(A_{1/n} - I)$. So $\{D_n\}$ is a bounded net. Let D be the weak* limit of the subnet D_{n_λ} of D_n. By a second application of Lemma 19.3[20], we have

$$\begin{aligned} \delta_D(T) &= \lim_{\lambda} (D_{n_\lambda} T - T D_{n_\lambda}) = \lim_{\lambda} n_\lambda (A_{1/n_\lambda} T - T A_{1/n_\lambda}) \\ &= \lim_{\lambda} n_\lambda (\phi_{1/n_\lambda} - id)(A) A_{1/\lambda} = \delta(T). \ \square \end{aligned}$$

Lemma 1.21[36]. *Let $\mathcal{M}$ be an infinite factor, and let $\mathcal{N}$ be a maximal nest of projections in $\mathcal{M}$. Then $\mathcal{N}$ contains either (perhaps both) an infinite increasing sequence $p_1 < p_2 < \ldots$ with the $\mathcal{N}$-intervals $p_{n+1} - p_n$ mutually equivalent in $\mathcal{M}$ or an infinite decreasing sequence $p_1 > p_2 > \ldots$ with the $\mathcal{N}$-intervals $p_n - p_{n+1}$ mutually equivalent in $\mathcal{M}$.*

Theorem 1.22 *Let $\mathcal{M}$ be a type II_∞ factor and $\mathcal{N}$ be a nest in $\mathcal{M}$. If δ is a derivation from $alg_{\mathcal{M}}\mathcal{N}$ into $\mathcal{M}$, then δ is inner.*

Proof. By Lemma 1.15, we may assume that $\mathcal{N}$ is a maximal nest in $\mathcal{M}$. By Lemma 1.21, we divide the proof into three cases.

Case 1. Let us first consider the case in which $\mathcal{N}$ contains an infinite sequence $0 = p_0 < p_1 < p_2 < ...$ with $p_n \to I$ in the strong operator topology such that $p_{n+1} - p_n$ are equivalent in $\mathcal{M}$. Let V_n in $\mathcal{M}$ such that $V_n^* V_n = p_{n+1} - p_n$ and $V_n V_n^* = p_n - p_{n-1}$. Let $V = \sum_{n=0}^{\infty} V_n$. Then V is in $alg_{\mathcal{M}}\mathcal{N}$ such that $VV^* = I$ and for any $A \in alg_{\mathcal{M}}\{p_0, p_2, ...\}$, AV and V^*AV^2 belong to $alg_{\mathcal{M}}\mathcal{N}$. Let $\mathcal{B} = alg_{\mathcal{M}}\{p_0, p_2, ...\}$. For any $A \in \mathcal{B}$, define $\overline{\delta}(A) = (\delta(AV) - A\delta(V))V^*$. In the following we will show that δ is a derivation from $\mathcal{B}$ into $\mathcal{M}$.

$$\begin{aligned}
\overline{\delta}(A)B + A\overline{\delta}(B) &= [\delta(AV) - A\delta(V)]V^*B + A[\delta(BV) - B\delta(V)]V^* \\
&= [\delta(AV) - A\delta(V)]V^*B \\
&\quad + A[\delta(BV) + BV\delta(V)V^* - BV\delta(V)V^* - B\delta(V)]V^* \\
&= [\delta(AV) - A\delta(V)]V^*BV^2V^{*2} + A[\delta(BV^2) - B\delta(V^2)]V^{*2} \\
&= [\delta(AV)V^*BV^{*2} + AV\delta(V^*BV^2)]V^{*2} - AB\delta(V^2)V^{*2} \\
&= [\delta(AVV^*BV^2) - AB\delta(V^2)]V^{*2} \\
&= [\delta(ABV^2) - AB\delta(V^2)]V^{*2} \\
&= \overline{\delta}(AB).
\end{aligned}$$

By Lemma 1.15, it follows that $\overline{\delta}$ is inner. For $A \in \mathcal{A}$ we have that $\overline{\delta}(A) = [\delta(A)V - A\delta(V) + A\delta(V)]V^* = \delta(A)$. Hence δ is inner.

Case 2. If the dual nest $\mathcal{N}^{\perp}$ satisfies the properties of the case 1, we consider that δ^*, where $\delta^*(A) = (\delta(A^*))^*$ for A in $\mathcal{A}^* = alg_M\mathcal{N}^{\perp}$. By case 1, we have δ^* is inner. Hence δ

is inner.

Case 3. Apply Lemma 1.17 yielding either $p_1 < p_2 < p_3 < \cdots$ or $p_1 > p_2 > \cdots$ with $p_{n+1} - p_n$ (resp. $p_n - p_{n+1}$) mutually equivalent in $\mathcal{M}$. By considering $\mathcal{N}^\perp$, we can assume that $p_n < p_{n+1}$. Let $p = \bigvee_{i=1}^{\infty} p_i$ and let $E = p - p_1$. Then $E\mathcal{N}E$ is a maximal nest in factor $E\mathcal{M}E$ and $E\mathcal{M}E$ is an infinite factor. Let $\delta_1(EAE) = E\delta(A)E$ for $A \in alg_{\mathcal{M}}\mathcal{N}$. Then δ_1 is a derivation from $alg_{E\mathcal{M}E}EAE$ into $E\mathcal{M}E$. By the cases (1) and (2), we have that δ_1 is inner. Since E is an infinite projection, by Lemma 1.17, we have that δ is inner. □

Theorem 1.23. *Let $\mathcal{M}$ be a type III factor and $\mathcal{N}$ be a nest in $\mathcal{M}$. If δ is a derivation from $alg_{\mathcal{M}}\mathcal{N}$ into $\mathcal{M}$, then δ is inner.*

Proof. By Lemma 1.14, we may assume that $\mathcal{N}$ is maximal in $\mathcal{M}$. Let $0 = p_0 < p_1 < p_2 < \cdots$ in $\mathcal{N}$ such that $p_n \to I$ in strong operator topology. Then $p_{n+1} - p_n$ is equivalent to $p_{m+1} - p_m$. Let $\mathcal{L} = \{0, p_1, p_2, ...\}$. Similar to the proof of case 1, we can construct a derivation $\overline{\delta}$ from $alg_{\mathcal{M}}\mathcal{L}$ into $\mathcal{M}$ such that $\overline{\delta}$ is inner and $\delta(A) = \overline{\delta}(A)$ for A in $alg_{\mathcal{M}}\mathcal{N}$. Hence δ is inner. □

Corollary 1.24. *Let $\mathcal{M}$ be a type II_∞ or type III factor and δ be a derivation from $alg_{\mathcal{M}}\mathcal{N}$ into every weakly closed $\mathcal{A}$-bimodule in $\mathcal{M}$ containing $\mathcal{A}$ is inner.*

3.2 Cohomology of certain operator algebras

Let $\mathcal{A}$ be a normed algebra over $\mathbf{C}$ and let M be a normed space over $\mathbf{C}$. M is said to be a *normed left* $\mathcal{A}$-module if M is a left $\mathcal{A}$-module, and there exists a positive constant K such that $||am|| \leq K||a||||m||$, whenever $m \in M$ and $a \in \mathcal{A}$. A similar definition applies to *normed right* $\mathcal{A}$-modules. A *normed* $\mathcal{A}$-module is an $\mathcal{A}$-bimodule that is both a normed left $\mathcal{A}$-module and a normed right $\mathcal{A}$-module. An $\mathcal{A}$-module M is said to be a dual $\mathcal{A}$-module if M is the dual of a normed space M_* and for each $a \in \mathcal{A}$ the mappings $m \to am, m \to ma : M \to M$ are weak* continuous. Let M be an $\mathcal{A}$-module and m in M, and let δ_m denote the mapping of $\mathcal{A}$ into M defined by $\delta_m(a) = am - ma$, for $a \in \mathcal{A}$. Then δ_m is a derivation from $\mathcal{A}$ into M. Each such derivation δ_m is called an inner derivation. If x is an invertible element in an algebra $\mathcal{A}$, then $ad(x)$ denotes the isomorphism $a \mapsto xax^{-1}$.

Let M be a $\mathcal{A}$-module. We denote by $C_c^n(\mathcal{A}, M)$ the linear space of all bounded n-linear mappings from $\mathcal{A} \times ... \times \mathcal{A}$ into M. The coboundary operator ∂, from $C_c^n(\mathcal{A}, M)$ into $C_c^{n+1}(\mathcal{A}, M)$, is defined by

$$\begin{aligned}(\partial\rho)(a_1, ...a_{n+1}) &= a_1\rho(a_2, a_3, ..., a_{n+1}) \\ &\quad + \sum_{i=1}^{n}(-1)^i\rho(a_1, ..., a_{i_1}, a_ia_{i+1}, ..., a_{n+1}) \\ &\quad + a_1\rho(a_1, a_2, ..., a_{n+})a_{n+1}.\end{aligned}$$

By convention, $C_c^0(\mathcal{A}, M) = M$, and $\rho : C_c^0(\mathcal{A}, M) \to C_c^1(\mathcal{A}, M)$ is defined by $(\partial m)(a) = am - ma$ for a in $\mathcal{A}$ and $m \in M$. For $n = 0, 1, 2...$, let $B_c^{n+1}(\mathcal{A}, M)$ denote the range of ∂ in $C_c^{n+1}(\mathcal{A}, M)$ and let $Z_c^n(\mathcal{A}, M)$ denote the nullspace of ∂ in $C_c^n(\mathcal{A}, M)$. It can be shown that $\partial^2 = 0$. The quotient space

$$Z_c^n(\mathcal{A}, M)/B_c^n(\mathcal{A}, M)$$

is denoted by $H_c^n(\mathcal{A}, M)$ and called the n-dimensional cohomology group (of $\mathcal{A}$ with coefficients in M). Let M be a dual normal $\mathcal{A}$-module. We denote by $C_w^n(\mathcal{A}, M)$ the linear space of all bounded n-linear mappings from $\mathcal{A} \times ... \times \mathcal{A}$ into M which are separately σ-weak to weak* continuous.

The (normal) coboundary operator, defined as before, provides a sequence of linear mappings

$$M = C_w^0(\mathcal{A}, M) \xrightarrow{\partial} C_w^1(\mathcal{A}, M) \xrightarrow{\partial} C_w^2(\mathcal{A}, M) \xrightarrow{\partial} ...$$

From this sequence, $B_w^n(\mathcal{A}, M)$, $Z_w^n(\mathcal{A}, M)$ and the cohomology group

$$H_w^n(\mathcal{A}, M) = Z_w^n(\mathcal{A}, M)/B_w^n(\mathcal{A}, M)$$

are defined just as in the case of norm continuous cohomology.

Let $\mathcal{N}$ be a nest in $B(H)$. In [64], Lance proves that $H_c^n(alg\mathcal{N}, B(H)) = 0$. In [34], Gilfeater, Hopenwasser and Larson improve this result and prove that if $\mathcal{L}$ is a finite-width commutative subspace lattice, then $H_c^n(alg\mathcal{L}, B(H)) = 0$. Nielsen[87] obtains that $H_c^n(alg\mathcal{N}, B(H)) = H_w^n(alg\mathcal{N}, B(H)) = 0$.

We will consider two cohomology theories, the norm continuous cohomology and the normal cohomology. We show that a class of non-selfadjoint operator algebras has isomorphic continuous and normal cohomology. This class contains reflexive algebras whose invariant subspace lattices are tensor products of nests and reflexive algebras with two-atom atomic Boolean subspace lattice.

3.2.1 Norm continuous cohomology

In this section, by using the technique in the proof of Theorem 2.1[64], we improve Theorem 2.1[64].

The proofs of the following Lemmas are easy. We leave them to the reader.

Lemma 2.1[92]. *Let $\mathcal{A}$ be a subalgebra of $B(H)$. Then $H_c^n(\mathcal{A}^*, B(H)) = H_c^n(\mathcal{A}, B(H))$ and $H_c^n(\mathcal{A}^*, \mathcal{A}^*) = H_c^n(\mathcal{A}, \mathcal{A})$.*

Lemma 2.2. *Let $\mathcal{A}$ be a subalgebra of $B(H)$ and let T be an invertible operator from a Hilbert space H into a Hilbert space K. Then $H_c^n(\mathcal{A}, B(H)) = H_c^n(T\mathcal{A}T^{-1}, B(K))$.*

Lemma 2.3. *Let $\mathcal{A}$ be a subalgebra of $B(H)$. Suppose that there exists $x \in H$, $x \neq 0$ such that for any $y \in H$, $x \otimes y \in \mathcal{A}$, then $H_c^n(\mathcal{A}, B(H)) = 0$.*

By Lemmas 2.1 to 2.3, we easily show the following result.

Corollary 2.4. *Let $\mathcal{A}$ be an operator algebra in $B(H)$. Let*

$$\mathcal{A}_1 = \left\{ \begin{pmatrix} * & 0 \\ * & A \end{pmatrix} : A \in \mathcal{A} \right\} \text{ on } \mathbf{C} \oplus H,$$

$$\mathcal{A}_2 = \left\{ \begin{pmatrix} * & * \\ 0 & A \end{pmatrix} : A \in \mathcal{A} \right\} \text{ on } \mathbf{C} \oplus H,$$

$$\mathcal{A}_3 = \left\{ \begin{pmatrix} A & * \\ 0 & * \end{pmatrix} : A \in \mathcal{A} \right\} \text{ on } H \oplus \mathbf{C},$$

$$\mathcal{A}_4 = \left\{ \begin{pmatrix} A & 0 \\ * & * \end{pmatrix} : A \in \mathcal{A} \right\} \text{ on } H \oplus \mathbf{C}.$$

Then $H_c^n(\mathcal{A}_i, B(\mathbf{C} \oplus H)) = 0$, for $i = 1, 2$ and $H_c^n(\mathcal{A}_i, B(H \oplus \mathbf{C})) = 0$, for $i = 3, 4$.

Lemma 2.5[54]. *If $\mathcal{A}$ is a selfadjoint operator algebra, $\mathcal{B}$ an operator algebra maximal with the property of having $\mathcal{A}$ as its intersection with its adjoint, N and M orthogonal projections with invariant under, and in $\mathcal{B}$, and B an operator such that $B = NBM$, then B lies in $\mathcal{B}$.*

Lemma 2.6. *Let $\mathcal{J}$ be a maximal triangular algebra on H and let $\mathcal{N} = lat\mathcal{J}$. Suppose that $E \in lat\mathcal{J}$ with $dim(E - E_-) \leq 1$. Then for any $x \in E$, $y \in (E_-)^\perp$, $x \otimes y \in \mathcal{J}$.*

Proof. Since

$$\begin{aligned} x \otimes y = E(x \otimes y)(I - E_-) &= E_-(x \otimes y)(I - E_-) \\ &\quad + (I - E_-)E(x \otimes y)(I - E) + (E - E_-)(x \otimes y)(E - E_-) \end{aligned}$$

by Lemma 2.5, it follows that $E_-(x \otimes y)(I - E_-)$ and $(I - E_-)E(x \otimes y)(I - E)$ belong to $\mathcal{J}$. Since $dim(E - E_-) \leq 1$, we have $(E - E_-)(x \otimes y)(E - E_-)$ is a scalar multiple of $E - E_-$. Hence $x \otimes y$ belongs to $\mathcal{J}$. □

Theorem 2.7. *Let $\mathcal{J}$ be a maximal triangular algebra on H and let $\mathcal{N} = lat\mathcal{J}$ with $dim(H - H_-) \leq 1$ or $din(0_+) \leq 1$. Then $H_c^n(\mathcal{J}, B(H)) = 0$.*

Proof. By the Lemma 2.1, we may assume that $dim(H - H_-) \leq 1$.

If $dim(H - H_-) = 1$, by Lemma 2.3, it follows that $H_c^n(\mathcal{J}, B(H)) = 0$.

Suppose $dim(H - H_-) = 0$. We can assume that $\mathcal{J}$ is norm closed. Choose $P_n \in lat\mathcal{J}$ with $P_n \to I$ in strong topology. For $y \in H$, choose $e_i \in P_i^\perp$ such that $||e_i|| = 1$. By Lemma 2.5, $P_i(y) \otimes e_i \in \mathcal{J}$.

For $\sigma \in Z_c^n(\mathcal{J}, B(H))$, define ϕ_i in $C_c^{n-1}(\mathcal{J}, B(H))$ by

$$\phi_i(a_1, ..., a_{n-1})y = (-1)^n \sigma(a_1, ..., a_{n-1}, P_i(y) \otimes e_i)e_i.$$

we can calculate $\partial\phi_i(a_1, ..., a_n)x = \sigma(a_1, ..., a_n)x$ for any $x \in P_i$. Since $\{\phi_i\}$ is bounded, we have that $\{\phi_i\}$ has a subset which converges in ultraweak topology to an element ϕ of $C_c^{n-1}(\mathcal{J}, B(H))$. A routine calculation gives $\sigma = \partial\phi$. □

3.2.2 Normal cohomology

In this section, We study normal cohomology on non-selfadjoint algebras. If $\mathcal{S}$ is a subset of $B(H)$, we denote $\mathcal{S}_1$ the unit ball of $\mathcal{S}$.

Theorem 2.8. *Let $\mathcal{A}$ be a subalgebra of $B(H)$ such that $\overline{(\mathcal{A} \cap K(H))_1} = (\ \overline{\mathcal{A}}\ )_1$ with $I \in \overline{\mathcal{A}}$. Then $H_c^n(\mathcal{A}, B(H)) = H_w^n(\mathcal{A}, B(H)) = H_w^n(\overline{\mathcal{A}}, B(H))$.*

Lemma 2.9. *Let $\mathcal{A}$ be a subalgebra of $B(H)$ such that $\overline{\mathcal{A}_1} = (\ \overline{\mathcal{A}}\ )_1$ and let τ be a bounded bilinear form on $\mathcal{A} \times \mathcal{A}$. If τ is separately σ-weakly continuous, then τ extends to a separately σ-weakly continuous bilinear form $\overline{\tau}$ on $\overline{\mathcal{A}} \times \mathcal{A}$.*

Proof. For a fixed $b \in \mathcal{A}$, we consider the σ-weakly continuous linear functional $T(b) : a \mapsto \tau(a, b)$ on $\mathcal{A}$. Let $T(b)$ extend to a σ-weakly continuous linear functional $S(b)$ on $\overline{\mathcal{A}}$. By $\overline{\mathcal{A}_1} = (\ \overline{\mathcal{A}}\ )_1$, it follows that such that $||S(b)|| \leq ||\tau||\,||b||$. Hence the mapping $S : b \mapsto S(b)$ is a bounded linear map from $\mathcal{A}$ into $(\ \overline{\mathcal{A}}\ )_*$ with $||S|| \leq ||\tau||$ and

$$< a, S(b) >= \tau(a, b), \text{ for } a,\ b \in \mathcal{A}, \tag{2.1}$$

where $< \cdot, \cdot >$ denotes the pairing between $\overline{\mathcal{A}}$ and its predual $(\ \overline{\mathcal{A}}\ )_*$.

Since τ is σ-weakly continuous in its second argument with the first fixed, (2.1) implies that S is a continuous linear map from $\mathcal{A}$ with the σ-weak topology into $(\ \overline{\mathcal{A}}\ )_*$ with the topology $\sigma((\ \overline{\mathcal{A}}\ )_*, \mathcal{A})$. Since S is a bounded map from $\mathcal{A}$ into $(\ \overline{\mathcal{A}}\ )_*$, it follows that $\mathcal{A}_1$ is relative σ-weakly compact in $\mathcal{A}$. Hence it follows that $S(\mathcal{A}_1)$ is relatively compact in $(\ \overline{\mathcal{A}}\ )_*$ with respect to the topology $\sigma((\ \overline{\mathcal{A}}\ )_*, \overline{\mathcal{A}})$; so this topology coincides, on $S(\mathcal{A}_1)$, with the Hausdorff topology $\sigma(\overline{\mathcal{A}}_*, \mathcal{A})$. Hence S is continuous as a mapping from $\mathcal{A}_1$, with the topology $\sigma(\mathcal{A}, (\overline{\mathcal{A}})_*)$ into $\overline{\mathcal{A}}_*$ with the topology $\sigma((\overline{\mathcal{A}})_*, \overline{\mathcal{A}})$.

Define $\overline{\tau}$ on $\overline{\mathcal{A}} \times \mathcal{A}$ by

$$\overline{\tau}(a, b) = < a, S(b) >$$

for all $a \in \overline{\mathcal{A}}$ and $b \in \mathcal{A}$; this $\overline{\tau}$ has the quired properties. □

Similarly, we can prove the following result.

Lemma 2.10. *Let $\mathcal{A}$ be as in Lemma 2.9 and let τ be a bounded n-linear form on $\mathcal{A} \times \cdots \times \mathcal{A}$. If τ is separately σ-weakly continuous, then τ extends to a separately σ-weakly continuous n-linear form $\overline{\tau}$ on $\overline{\mathcal{A}} \times \cdots \times \overline{\mathcal{A}}$.*

By Lemma 2.10, we have

Lemma 2.11. *Let $\mathcal{A}$ be as in Lemma 2.9. Then $H^n_w(\mathcal{A}, B(H)) = H^n_w(\overline{\mathcal{A}}, B(H))$.*

An argument similar to the proof of Theorem 5.3[98] yields

Lemma 2.12. *Let $\mathcal{A}$ as in Lemma 2.9 and let M be the dual of a Banach space M_*. If ϕ is a bounded n-linear map from $\mathcal{A} \times \cdots \times \mathcal{A}$ into M that is separately continuous relative to the σ-weak topology on $\mathcal{A}$ and the weak* topology on M, then ϕ extends to a bounded n-linear map $\overline{\phi}$ from $\overline{\mathcal{A}} \times \cdots \times \overline{\mathcal{A}}$ into M, which is separately continuous relative to the σ-weak topology on $\overline{\mathcal{A}}$ and weak* topology on M.*

Let $\mathcal{A}$ be a subalgebra of $B(H)$ and let Λ the set of all singular states on $B(H)$. For each $f \in \Lambda$, $\{\pi_f, H_f\}$ denotes the GNS-construction of $B(H)$ with respect to f. Let $\hat{H} = \sum_{f \in \Lambda} \oplus H_f$ and $\eta : B(H) \to B(\hat{H})$ be the $*$-homomorphism defined by

$$\eta(x)(\sum_{f \in \Lambda} \oplus \xi_f) = \sum_{f \in \Phi} \oplus \pi_f(x)\xi_f.$$

Let $K = \hat{H} \oplus H$. Define $\pi : B(H) \to B(K)$ by

$$\pi(x) = \begin{pmatrix} \eta(x) & 0 \\ 0 & x \end{pmatrix}$$

Then $\pi(x)$ is a faithful $*$-representation of $B(H)$.

Lemma 2.13. *Let $\mathcal{A}$ be a subalgebra of $B(H)$ and let ϕ be a bounded linear functional on $\pi(\mathcal{A})$. Then there exists a σ-weakly continuous functional ψ on $B(K)$ such that $\phi = \psi|_{\pi(\mathcal{A})}$.*

Proof. Let $\hat{\phi} = \phi \circ \pi$. Since π is isometric, it follows that $\hat{\phi}$ is a bounded functional on $\mathcal{A}$. Extend $\hat{\phi}$ to a bounded functional ω on $B(H)$ such that $||\omega|| = ||\hat{\phi}||$.

Define $\psi_1(\pi(x)) = \omega(x)$ for $x \in B(H)$. Since π is an isometry, it follows that ψ_1 is well-defined and $\psi_1 \in (\pi(B(H)))^*$. Let $\omega = f + g$, such that f is singular and $g \in B(H)_*$. We can prove that ψ_1 is σ-weakly continuous on $\pi(B(H))$. Extending ψ_1 to a σ-weakly continuous functional ψ on $B(K)$, it is clear that ψ extends ϕ. □

Suppose $\mathcal{A}$ is as in Theorem 2.8. Define

$$P = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}$$

we easily show that $P \in \overline{\pi(\mathcal{A})} \cap (\overline{\pi(\mathcal{A})})'$. Define $\alpha : \overline{\pi(\mathcal{A})}P \to \overline{\mathcal{A}}$ by

$$\begin{pmatrix} 0 & 0 \\ 0 & x \end{pmatrix} \mapsto x.$$

The α is an isometric algebraic isomorphism, and is σ-weakly continuous. If $x \in \mathcal{A}$, then $\alpha(\pi(x)P) \in \mathcal{A}$.

Lemma 2.14. *Let $\mathcal{A}$ be as in Theorem 2.8. For $n \geq 1$, let $\rho \in Z_c^n(\mathcal{A}, B(H))$. Then there exists $\xi \in C_c^{n-1}(\mathcal{A}, B(H))$ such that $\rho - \partial\xi \in Z_w^n(\mathcal{A}, B(H))$.*

Proof. If $A \in \overline{\pi(\mathcal{A})}$, then $\alpha(AP) \in \overline{\mathcal{A}}$. So we can define left and right actions of $\overline{\pi(\mathcal{A})}$ on $B(H)$ by $A \cdot m = \alpha(AP)m$, $m \cdot A = m\alpha(AP), m \in B(H)$. By the definition, $B(H)$ become a $\overline{\pi(\mathcal{A})}$-module such that

$$P \cdot m = m \cdot P = m, \tag{2.2}$$

for any $m \in B(H)$. For $\rho \in Z_c^n(\mathcal{A}, B(H))$, since α is isometric, we can define ρ_1 in $C_c^n(\pi(\mathcal{A}), B(H))$ by

$$\rho_1(A_1, \cdots, A_n) = \rho(\alpha(A_1P), \cdots, \alpha(A_nP)),$$

for all $A_1, ..., A_n \in \pi(\mathcal{A})$. A routine calculation gives

$$(\partial\rho_1)(A_1, ..., A_{n+1}) = (\partial\rho)(\alpha(A_1P), ..., \alpha(A_{n+1}P))$$

for all $A_1, ..., A_{n+1}$ in $\pi(\mathcal{A})$; so $\rho_1 \in Z_c^n(\pi(\mathcal{A}), B(H))$. Let $\eta \in B(H)_*$, A, $A_i \in \pi(\mathcal{A})$, $1 \leq j \leq n$, and define

$$\phi_j(A) = \eta(\rho_1(A_1, ..., A_{j-1}, A, A_{j+1}, ..., A_n),$$

for $1 \leq j \leq n$, By Lemma 2.13, each ϕ_j is the restriction of an σ-weak continuous functional on $B(K)$, so it is σ-weak continuous on $\pi(\mathcal{A})$. This proves that ρ_1 is separately σ-weak to weak* continuous. By Lemma 2.12, ρ_1 extends to a bounded n-linear mapping $\overline{\rho}_1$: $\overline{\pi(\mathcal{A})} \times \cdots \times \overline{\pi(\mathcal{A})} \to B(H)$ which is also separately σ-weak to weak* continuous. It follows from Lemma 6.2[98] that there exists ξ_1 in $C_w^n(\overline{\pi(\mathcal{A})}, B(H))$ such that $\overline{\rho}_1 - \partial\xi_1$ vanishes whenever any of its arguments lies in the $span\{I, 2P - I\}$. Since $P \in span\{I, 2P - I\}$, it follows from (2.2) that

$$(\overline{\rho}_1 - \partial\xi_1)(A_1, ..., A_n) = P(\overline{\rho}_1 - \partial\xi_1)(A_1, ..., A_n) = (\overline{\rho}_1 - \partial\xi_1)(PA_1, ..., PA_n)$$

for all $A_1, ..., A_n$ in $\overline{\pi(\mathcal{A})}$. Now we define

$$\xi(\alpha(A_1 P), ..., \alpha(A_{n-1} P)) = \xi_1(A_1, ..., A_{n-1})$$

whenever $A_1, ..., A_{n-1} \in \pi(\mathcal{A})$. A routine calculation shows that $\rho - \partial\xi \in Z_w^n(\mathcal{A}, B(H))$. □

By a routine modification of the proof of Lemma 6.5[98], we can prove the following result.

Lemma 2.15. *Let $\mathcal{A}$ be as in Theorem 2.8. Then $B_c^n(\mathcal{A},\ B(H)) \cap Z_w^n(\mathcal{A},\ B(H)) = B_w^n(\mathcal{A},\ B(H))$.*

The Proof of Theorem 2.8.

For each ρ in $Z_w^n(\mathcal{A}, B(H))$, the coset $\rho + B_w^n(\mathcal{A}, B(H))$ is a subset of the coset $\rho + B_c^n(\mathcal{A}, B(H))$. Hence there is a natural homomorphism

$$\Phi: \ \rho + B_w^n(\mathcal{A}, B(H)) \to \rho + B_c^n(\mathcal{A}, B(H))$$

from $H_w^n(\mathcal{A}, B(H))$ into $H_c^n(\mathcal{A}, B(H))$. By Lemma 2.11, Φ is injective. By Lemma 2.15, the range of Φ is $H_c^n(\mathcal{A}, B(H))$. Hence $H_c^n(\mathcal{A}, B(H)) \cong H_w^n(\mathcal{A}, B(H))$. By Lemma 2.15, we have $H_c^n(\mathcal{A}, B(H)) = H_w^n(\mathcal{A}, B(H)) = H_w^n(\overline{\mathcal{A}}, B(H))$. □

Corollary 2.16. *Let $\mathcal{L} = \mathcal{N}_1 \otimes ... \otimes \mathcal{N}_n$, where $\mathcal{N}_i$ are nests on H_i and let $\mathcal{A}$ be any subalgebra of alg$\mathcal{L}$ containing all rank-one operators of alg$\mathcal{L}$. Then $H_w^n(\mathcal{A}, B(H)) = H_c^n(\mathcal{A}, B(H)) = 0$.*

Proof. By Theorem 2.6[34], we have that $\mathcal{A}$ satisfies the condition of Theorem 2.8. By $\mathcal{L}$ is finite-width and Theorem 3.1[34], it follows that $H_w^n(\mathcal{A}, B(H)) = H_c^n(\mathcal{A}, B(H)) = H_w^n(\overline{\mathcal{A}}, B(H)) = 0$. □

Corollary 2.17. *Let $\mathcal{L}$ be an atomic Boolean subspace lattice with two atoms and let $\mathcal{A}$*

be any subalgebra of alg$\mathcal{L}$ containing all rank-one operators of alg$\mathcal{L}$. Then $H_w^n(\mathcal{A}, B(H)) = H_c^n(\mathcal{A}, B(H))$.

Corollary 2.18. *Let $\mathcal{A}$ be a strongly reducible maximal triangular algebra on H. Then* $H_c^n(\mathcal{A}, B(H)) = 0$.

REFERENCES

1. M. Anoussis, A. Katavolos and M. Lambrou, *On the reflexive algebra with two invariant subspaces*, J. Operator Theory **30** (1993), 267-299.
2. R. Ara and M. Mathieu, *On the central Haagerup tensor product*, Proc. Edinburgh Math. Soc. (2) **37** (1994), 161-174.
3. R. Archbold, M. Mathieu amd D. Somerset, *Elementary operators on antiliminal C^*-algebras*, Math. Ann. **313** (1999), 609-616.
4. S. Argyros, M. S. Lambrou and W. Longstaff, *Atomic Boolean Subspace Lattices and Applications to the Theory of Bases*, Memoirs Amer. Math. Soc. **445** (1991).
5. W. Arveson, *An invitation to C^*-algebra*, Springer-Verlag, New York, Heidelberg, Berlin, 1976.
6. W. Arveson, *Operator algebras and invariant subspaces*, Ann. Math **100** (1977), 433-532.
7. B. Aupetit, *An improvement of Kaplansky's lemma on locally algebraic operators*, Studia Math **88** (1988), 275-278.
8. E. Azoff, *On finite rank operators and preannihilators*, Memoir Amer. Math. Soc. **357** (1986).
9. E. Azoff, L. Ding and W. Wogen, *Separating versus strictly separating vectors*, Proc. Amer. Math. Soc. **124** (1996), 3135-3142.
10. E. Azoff and H. Shehada, *Algebras generated by mutually orthogonal idempotent operators*, J. Operator Theory **29** (1993), 249–267.
11. E. Azoff and H. Shehada, *Literal embeddings of linear spaces of operators*, Indiana Univ. Math. J. **42** (1993), 571–589.
12. C. Batty, *Irreducible representations of inseparable C^*-algebras*, Rocky Mountain J. **14** (1984), 721-727.
13. F. Bonsall and J. Duncan, *Complete Normed Algebras*, Springer-Verlag, New York-Heidelberg, 1973.
14. S. Bradley, *Finite rank operators in certain algebras*, Canad. Math. Bull. **42** (1999), 452-462.
15. P. R. Chernoff, *Representations, automorphisms, and derivations of some operator algebras*, J. Funct. Anal **12** (1973), 275-289.
16. M. D. Choi, *Positive linear maps on C^*-algebras*, Canad. J. Math. **24** (1972), 520-529.
17. M. D. Choi, *Completely positive linear maps on complex matrices*, Linear Algebra and Appl. **10** (1975), 285–290.
18. E. Christensend, *Derivations of nest algebras*, Math. Ann. **229** (1977), 155-161.
19. E. Christensend and C. Peligrad, *Commutants of nest algebras modulo the compact operators*, Inv. Math. **56** (1980), 113-116.
20. K. R. Davidson, *Nest Algebras*, Pittman Research Notes in Mathematics Series 191, 1988.
21. Lifeng Ding, *An algebraic reflexivity result*, Houston J. Math. **19** (1993), 533-540.

22. Lifeng Ding, *On a pattern of reflexive operator spaces*, Proc. Amer. Math. Soc. **124** (1996), 3101–3108.
23. Lifeng Ding, *Separating vectors and reflexivity*, Linear Algebra Appl **174** (1992), 37–52.
24. Jacques Dixmier, *von Neumann algebras*, North-Holland Mathematical Library, 27. North-Holland Publishing Co., Amsterdam-New York, 1981.
25. Hongke Du and Jianhua Zhang, *Derivations on nest subalgebras of von Neumann algebras*, Chinese Ann. Math. Ser. (4) **17** (1996), 467-474.
26. Hongke Du and Jianhua Zhang, *Derivations on nest subalgebras of von Neumann algebras II*, Acta Math. Sinica (3) **40** (1997), 357-362.
27. J. A. Erdos, *Some results on triangular operator algebras*, Amer. J. Math. **89** (1967), 85-93..
28. J. A. Erdos, *On some ideals of nest algebras*, Proc. London Math. Soc. **44** (1982), 143-160.
29. J. A. Erdos, *Reflexivity for subspace maps and linear spaces of operators*, Proc. London Math. Soc. **53** (1986), 582-600.
30. J. A. Erdos and S. C. Power, *Weakly closed ideals of nest algebras*, J. Operator Theory **7** (1982), 219-235.
31. J. Fell, *The structure of algebras of operator fields*, Acta Math. **106** (1961), 233-280.
32. F. Gilfeather, *Derivations on certain CSL algebras*, J. Operator Theory **11** (1984), 145-156.
33. F. Gilfeather, *Derivations on certain CSL algebras. II*, Internat. Schriftenreihe Numer. Math., 65, Birkhuser, Basel, (1984), 135–140.
34. F. Gilfeather, A. Hopenwasser and D. Larson, *Reflexive algebras with finite width lattices: tensor products, cohomology, compact perturbations*, J. Func Anal **55** (1984), 176-199.
35. F. Gilfeather and D. Larson, *Nest subalgebras of von Neumann algebras*, Adv. Math. **46** (1982), 171-199.
36. F. Gilfeather and D. Larson, *Nest subalgebras of von Neumann algebras: commutants modulo compacts and distance estimates*, J. Operator Theory **7** (1982), 279-302.
37. F. Gilfeather and R. L. Moore, *Isomorphisms of certain CSL algebras*, J. Funct. Anal. **67** (1986), 264-291.
38. F. Gilfeather and R. Smith, *Cohomology for operator algebras: cones and suspensions*, Proc. London Math. **65** (1992), 175-180.
39. W. Gong, D. Larson and W. Wogen, *Two results on separating vectors*, Indiana Univ. Math. J. **43** (1994), 1159–1165.
40. Don Hadwin, *A general view of reflexivity*, Trans. Amer. Math. Soc. **344** (1994), 325–360.
41. Don Hadwin, *Algebraically reflexive linear transformations*, Linear and Multilinear Algebra **14** (1983), 225–233.
42. Don Hadwin and D. Han, *Weak reflexivity of operators*, preprint.
43. Don Hadwin and E. Nordgren, *Subalgebras of reflexive algebras*, J. Operator Theory **7** (1982), 3-23.
44. P. R. Halmos, *Reflexive lattices of subspaces*, J. London Mat. Soc. **4** (1971), 257-263.
45. D. Han, *On A-submodules for reflexive operator algebras*, Proc. Amer. Math. Soc. **104** (1988), 1067-1070.

46. K. J. Harrison and U. A. Mueller, *Decomposability of reflexive cycle algebras*, J. London Math. Soc. **51** (1995), 148-160.

47. A. Hopenwasser, *Complete distributivity*, Proc. Sympos. Pure Math., 51, Part 1, Amer. Math. Soc., Providence, RI, 1990, 285–305.

48. A. Hopenwasser and R. Moore, *Finite rank operators in reflexive operators algebras*, J. London Math. Soc. **27** (1983), 331-338.

49. J. Hou, *Linear interpolations and elementary operators on $B(H)$*, Sci. China, Ser. A **36** (1993), 1025-1035.

50. J. Hou, *On operator inequalities and linear combinations of operators*, Linear Algebra Appl. **153** (1991), 35–51.

51. B. E. Johnson, *Cohomology in Banach algebras*, Mem. Amer. Math. Soc. **127** (1972).

52. B. E. Jonson, R. Kadison and J. Rringrose, *Cohomology of operator algebras III. Reduction to normal cohomology*, Bull. Soc. Math. France **100** (1972), 1973-96.

53. R. Kadison and J. Ringrose, *Fundamentals of the Operator Algebras*, vols. I and II Academic Pree, Orlando, 1983 and 1986.

54. R. V. Kadison and I. M. Singer, *Triangular operator algebras*, Amer. J. Math. **82** (1960), 227-259..

55. A. Katavilod and E. Katsoulis, *Semisimplicity in operator algebras and subspaces*, J. London Math. Soc. **42** (1990), 365-372.

56. A. Katavolos, M. S. Lambrou and W. E. Logstaff, *The decomposability of operators relative to two subspaces*, Studia Math. **105** (1993), 25-36.

57. A. Katavolos and M. Lambrou and M. Papadakis, *On some algebras diagonalized by M-bases*, Integral Equations Operator Theory **17** (1993), 68-93.

58. G. J. Knowles, *C_p perturbations of nest algebras*, Proc. Amer. Math. Soc. **92** (1984), 37-40.

59. G. J. Knowles, *Quotients of nest algebras with trivial commutator*, Pacific J. Math. **127** (1987), 121-126.

60. J. Kraus and D. R. Larson, *Some applications of a technique for constructing reflexive operator algebras*, J. Operator Theory **13** (1985), 227-236.

61. J. Kraus and D. R. Larson, *Reflexivity and distance formulae*, Proc. London Math. Soc. (2) **53** (1986), 340-356.

62. M Lambrou, *Approximants, commutants and double commutants in normed algebras*, J. London Math. Soc. **25** (1982), 499-512.

63. M. S. Lambrou, *On the rank of operators in reflexive algebras*, Linear Algebra Appl. **142** (1990), 211-235.

64. E. Lance, *Cohomology and perturbations of nest algebras*, Proc. London Math. Soc. **43** (1981), 334-356.

65. D. Larson, *Annihilators of operator algebras*, Operator Theory: Adv. Appl. **6** (1982), 119-130.

66. D. R. Larson, *Hyperreflexivity and a dual product construction*, Trans. Amer. Math. Soc. **294** (1986), 79-88.

67. D. Larson, *Reflexivity, algebraic reflexivity and linear interpolation*, Amer. J. Math. Soc. **110** (1988), 283–299.

68. D. Larson and B. Solel, *Nests and inner flows*, J. Operator Theory **16** (1986), 157-164.

69. D. R. Larson and A. D. Sourour, *Local derivations and local automorphisms of $B(X)$*, Proc. Sympos. Pure Math. (2) **51** (1990), 187-194..

70. Jiankui Li, *Complete positivity of elementary operators*, Proc. Amer. Math. Soc, **127** (1999), 235-239.

71. Jiankui Li, *Elementary operators and completely positive linear maps*, Northeast. Math. J. (3) **11** (1995), 349-354.

72. Jiankui Li, *Commutants and double commutants of reflexive algebras*, Kyushu J. Math. **50** (1996), 171-178.

73. Jiankui Li, *Some properties of pentagon subspace lattices*, J. Math. Res. Exposition **17** (1997), 226-270.

74. Jiankui Li, *Complete positivity of elementary operators on C^*-algebras*, Chinese Quart. J. Math. (4) **12** (1997), 51-54.

75. Jiankui Li, *Decomposability of certain reflexive algebras*, Houston J. Math **23** (1997), 121-126.

76. Jiankui Li, *A remark on complete positivity of elementary operators*, Integral Equations Operator Theory **28** (1997), 110–115.

77. Jiankui Li and Hemant Pendharkar, *Derivations of certain operator algebras*, Int. J. Math. Math. Sci.(5) **24** (2000), 345-349.

78. Jiankui Li and Zhidong Pan, *Reflexivity of a finite dimensional subspace of operators*, J. Operator Theory, to appear.

79. A. I. Loginov and G. S. Sulman, *Hereditary and intermediate reflexivity of W^*-algebras*, Math USSR-Izv **9** (1975), 1189-1201.

80. W. E. Longstaff, *Strongly reflexive lattices*, J. London Math. Soc. **11** (1975), 491-498.

81. W. E. Longstaff, *A note on the semisimplcity of reflexive operator algebras*, Proc. Int. Workshop on Analysis and its Applic., 4th Annual Meeting (Dubrovnik-Kupairi, Yugosalavia, 1990), 45-50.

82. W. Longstaff, *Remarks on semi-simple reflexive algebras*, Proc. Conf. Automatic Continuity and Banach Algebras, R. J. Loy (ed.), Centr Math. Anal. 21, Austral. Nat. Univ., Canberra, 1989, 273-287.

83. W. Longstaff, J. Nation and O. Panaia, *Abstract reflexive subspaces and completely distributive collapsibility*, Bull. Austral. Math. Soc. **58** (1998), 245-260.

84. W. Longstaff and O. Panaia, *$\mathcal{J}$-subspaces and subspace M-bases*, Studia Math. **139** (2000), 197-212.

85. B. Magajna, *On the relative reflexivity of finitely generated modules of operators*, Trans. Amer. Math. Soc. **327** (1991), 221-249.

86. M. Mathieu, *Elementary operators on prime C^*-algebras I*, Math. Ann. **284** (1989), 223-244.

87. J. Nielsen, *Cohomology of some non-selfadjoint operator algebras*, Math. Scand. **47** (1980), 150-156.

88. John Lindsay Orr, *Triangular algebras and ideals of nest algebras*, Memoirs Amer. Math. Soc. **562** (1995).
89. O. Panaia, *Algebraic isomorphisms and finite distributive subspace lattices*, J. London Math. Soc. **59** (1999), 1033-1045.
90. O. Panaia, *Quasi-spatiality of isomorphisms for certain classes of operator algebras*, Ph.D. Dissertation, University of Western Australia, 1995.
91. G. Pedersen, *C^*-algebras and their automorphism groups*, Academic Press, London, 1979.
92. D. Pitts, *Perturbations of certain reflexive algebras*, Pacific J. Math. **165** (1994), 161-180.
93. V. Plausen, *Completely bounded maps and dilations*, Pitman Research Notes in Mathematics Series, 146. New York, 1986.
94. F. Pop, *Derivations of certain nest-algebras of von Neumann algebras*, Oper. Theory Adv. Appl. **43** (1990), 279-288.
95. H. Radjavi and and O. Rosenthal, *Invariant Subspaces*, Springer-Verlag, New York-Heidelberg, 1973.
96. R. J. Ringrose, *On some algebras of operators*, Proc. London Math. Soc. **15** (1965), 61-83.
97. R. J. Ringrose, *On some algebras of operators II*, Proc. London Math. Soc. **16** (1966), 385-402.
98. J. R. Ringrose, *Coholomogy of operator algebras*, Lecture Notes in Math., vol 247, Springer-Verlag, New York, 1972, pp355-433.
99. S. Sakai, *Derivations of W^*-algebras*, Ann. Math. **83** (1966), 273-279.
100. A. Sinclair and R. Smith, *Hochschild Cohology of von Neumann Algebras*, London Math. Soc. Lecture Notes Ser., vol 203, Cambridge Press, Cambridge, 1995.
101. R. Smith, *Completely bounded maps between C^*-algebras*, J. London Math. Soc. **27** (1983), 157-166.
102. R. Smith, *Elementary operators and Haagerup tensor product*, Elementary operators and applications, 233–241, World Sci. Publishing, River Edge, NJ, 1992.
103. D. Somerset, *The central Haagerup tensor product of a C^*-algebras*, J. Operator Theory **39** (1998), 113-121.
104. R. Timoney, *A note on positivity of elementary operators*, Bull. London Math. Soc. **32** (2000), 229–234.
105. J. Tomiyama, *Recent development of the theory of completely bounded maps between C^*-algebras*, Publ. RIMS, Kyoto Univ. **19** (1983), 1283-1303.
106. J. Tomiyama, *On the difference of n-positive and complete positivity*, J. Funct. Anal. **49** (1982), 1-9.
107. J. Tomiyama, *On the transpose map of matrix algebras*, Proc. Amer. Math. Soc. **88** (1983), 635-638.
108. W. Wogen, *Some counterexamples in nonselfadjoint algebras*, Ann. of Math. **124** (1987), 415-427.